1 Introduction

1.1 Aims of the course

A large number of different phenomena in the natural world involve wave motion, as was demonstrated in "Waves in Physics" at Level 1. By studying the Mathematics of the Wave Equation, we can "simultaneously" explain many diverse phenomena. I will be talking a lot about problems that are "analogues" of each other and an important element of the course will be to make connections between seemingly different areas of Physics.

We will start off describing four different types of waves. From there, we will look in the abstract at some properties that are characteristic of wave motion in general before going on to apply our ideas to some very concrete examples that we come across in everyday life.

1.2 Recommended textbooks

Three textbooks will be particularly useful for this course:


Of these three books, I would regard Pain as an essential for the course — you really should buy this one, as much of the course can be found, explained slightly differently, here. The Feynman Lectures are a real classic, by Nobel Prize Winner Richard Feynman, who was also a gifted teacher.
and real character. They contain a lot of useful material for many different courses, explained at an appropriate level for 2nd Year use. Above all, they have a very informal chatty style — definitely worth thinking about, as they will help you with other courses apart from this one. Boas will help you with any mathematical difficulties you might have with the course. If you haven’t got it, then use the library copy for reference in the case of specific problems.

1.3 Coursework

This consists of a list of five questions. Of these, the first four are straightforward pencil and paper questions, dealing with various aspects of the course, but the last is really a mini-project. It involves writing and documenting a computer program to demonstrate an aspect of Fourier analysis and it parallels an experiment called “Transmission Lines” that you should all be doing in the 2EM lab. The deadline for handing in these questions will be Friday, 14th March, i.e., the Friday of Week 8, two weeks after the end of the course. I will be available for discussion on the mini-project outside formal lecture time.

2 Examples of Wave Propagation

2.1 Introduction

We will start the course by analysing some examples of wave motion. Some of these will be familiar from “A”-level or from previous courses, but some will be new. The important thing to notice is the mathematical similarity of the results we obtain.
2.2 Waves on a string

This material should be largely a revision of 1WP.

Consider a stretched string with density (i.e., mass per unit length) \( \rho \) and under tension \( T \) — see Fig. 1. The motion of an individual length element of the string is in the \( y \)-direction (i.e., we are looking at transverse waves). We use Newton’s 2nd Law “\( F = ma \)” to look at the vertical acceleration:

\[
\text{Net force} = T \sin[\theta(x + \Delta x)] - T \sin[\theta(x)] .
\]  

(1)

We make the important approximation that all our displacements are small. Life gets a little more complicated otherwise — see section 3.3. If \( y \) is small, then \( \theta \) is small and so \( \sin \theta \approx \tan \theta = (\partial y / \partial x) \). (Notice at this point the use of the partial derivative, since \( y \) varies with both \( x \) and \( t \).) Hence

\[
\text{Net force} = T \left[ \frac{\partial y}{\partial x} \right]_{x+\Delta x} - \left[ \frac{\partial y}{\partial x} \right]_x \\
\approx T \left( \frac{\partial^2 y}{\partial x^2} \right) \Delta x .
\]  

(2)

If we now equate this force to the mass that is being moved, which is \( \rho \Delta x \) and take the limit as \( \Delta x \to 0 \), we obtain

\[
T \left( \frac{\partial^2 y}{\partial x^2} \right) dx = (\rho dx) \left( \frac{\partial^2 y}{\partial t^2} \right) \Rightarrow \frac{\partial^2 y}{\partial x^2} = \frac{\rho}{T} \frac{\partial^2 y}{\partial t^2} .
\]  

(3)
We identify this as the Wave Equation and notice that we can rewrite it as

\[
\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 y}{\partial t^2},
\]  

where \( c \) is the wave propagation velocity.

### 2.3 Sound waves in a gas

In a gas, transverse waves, as described above, do not propagate, because we do not have a force analogous to the tension in the string that holds the particles together. Instead, there are compressional forces stopping the particles coming too close. As we shall see, this gives rise to "compression" or "longitudinal" waves — see Fig. 2. Here, I will give just a sketch derivation — full details are in Pain p. 160ff and Feynman I-47-4.

There are two key physical effects:

- When the gas moves, it creates density and pressure fluctuations, which are linked;

- Inequalities in pressure cause further motion of the gas.

Figure 2: Particle motion in compression waves
Figure 3: Definition of variables for compression waves in a gas

As with case of the vibrating string, we need to make some approximations:

- All pressure variations are small compared with atmospheric pressure;
- The size of the regions of compression and rarefaction (i.e., $\lambda$) are much larger than the mean free paths of the individual gas molecules.

The first approximation means that we are operating in what is called the linear regime — see Sec. 3.3 for further details. The second allows us to regard the gas as a continuous medium with a pressure and density that vary with $x$ and $t$. We don’t need to worry about the details of individual molecular collisions, but can replace their combined effect by a few simple constants that are experimentally measurable for a given gas.

$$P(x, t) = P_0 + p(x, t) \quad \rho(x, t) = \rho_0(x, t) + \rho_e(x, t).$$ \hspace{1cm} (5)

Here, $P_0$ is atmospheric pressure and $p$ is the excess pressure in the compressions (it will be negative in rarefactions) and is often termed the “acoustic pressure”. $\rho_0$ and $\rho_e$ are the corresponding densities. Note that from the first approximation above, $p \ll P_0$ and $\rho_e \ll \rho_0$.

When a gas (e.g., air) carries a plane wave in the $x$-direction, we can imagine each slab of air at a point $x$ in the undisturbed medium as moving to a new point $x + \eta(x, t)$, as shown in Fig. 3. At the same time, the volume
Figure 4: Motion of the gas is caused by unequal pressures on opposite sides of a “slab” of gas and we can apply Newton’s 2nd Law.

of this slab of air changes and, hence, so do the density and pressure. If we look at a surface $A$,

$$\text{Volume} \quad A\Delta x \quad \rightarrow \quad A[\Delta x + \eta(x + \Delta x, t) - \eta(x, t)]$$

$$= \quad A \left[ \Delta x + \frac{\partial \eta}{\partial x} \Delta x \right]$$

$$\text{(6)}$$

$$\text{Density} \quad \frac{\Delta m}{A\Delta x} \quad \rightarrow \quad \frac{\Delta m}{A[\Delta x + \frac{\partial \eta}{\partial x} \Delta x]} \quad \text{or} \quad \rho_0 \rightarrow \frac{\rho_0}{1 + \frac{\partial \eta}{\partial x}}.$$}

As long as our density fluctuations are small, we can use the binomial expansion:

$$\rho_0 \left[ 1 + \left( \frac{\partial \eta}{\partial x} \right) \right]^{-1} \approx \rho_0 \left[ 1 - \left( \frac{\partial \eta}{\partial x} \right) + \ldots \right]$$

$$\text{(7)}$$

and we identify $\rho_e = -\rho_0 \left( \frac{\partial \eta}{\partial x} \right)$ as the excess density. Now it turns out that, to a good approximation, there is a simple relationship between the pressure and density near atmospheric pressure, $p = B_0 \rho_e/\rho_0$, where $B_0$ is simply a material property of the gas, known as the bulk modulus. So we now also have the relation between the pressure and the particle displacement, namely

$$p = -B_0 \left( \frac{\partial \eta}{\partial x} \right).$$

$$\text{(8)}$$
2 EXAMPLES OF WAVE PROPAGATION

We now apply Newton’s 2nd Law of motion exactly as with the string. The motion of a small mass of gas is caused by unequal pressures on the opposite sides, as in Fig. 4. The net force on the element of mass $\Delta m$ is

$$P(x, t)A - P(x + \Delta x, t)A = A[P_0 + p(x, t) - P_0 + p(x + \Delta x, t)]$$  \hspace{1cm} (9)

$$\approx -A\left(\frac{\partial p}{\partial x}\right)\Delta x$$  \hspace{1cm} (10)

$$\approx AB_0\left(\frac{\partial^2 \eta}{\partial x^2}\right)\Delta x.$$  \hspace{1cm} (11)

So now we have the “$F$” bit of $F = ma$ and the $ma$ bit is simply

$$\Delta m\left(\frac{\partial^2 \eta}{\partial t^2}\right) = \rho A\Delta x\left(\frac{\partial^2 \eta}{\partial x^2}\right).$$  \hspace{1cm} (12)

It turns out that, to first order, we can ignore the difference between $\rho$ and $\rho_0$ in this equation to give, finally

$$\frac{\partial^2 \eta}{\partial x^2} = \frac{\rho_0}{B_0} \frac{\partial^2 \eta}{\partial t^2} \quad \text{or} \quad \frac{\partial^2 \eta}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 \eta}{\partial t^2},$$  \hspace{1cm} (13)

i.e., the wave equation, with propagation speed $c = \sqrt{B_0/\rho_0}$

2.4 Waves in transmission lines

More details on this subject can be found in Pain p. 179ff.

Consider a simplistic view of a transmission line, as simply two parallel wires lying next to each other (Fig. 5). If we place a voltage across one end, we might naïvely expect an equal voltage to appear across the load at the other end immediately.

However, this is not the case. We know from relativity courses that no signal can be transmitted faster than the speed of light, so at the very least,
it must take a finite length of time for the load to “see” the source. Further clues to what happens lie in the way that electrons in the wire feel both electric and magnetic effects from both the same wire and the one parallel to it. These effects are related to the capacitance and inductance of the transmission line — see Fig. 6.

Both the capacitance of the line and its inductance are proportional to the length of the line. This means that the important parameters are the capacitance per unit length $C_0$ and the inductance per unit length $L_0$. In particular, this allows us to make a revised “circuit diagram” of the transmission line as a whole, redrawing it in terms of a set of capacitors and inductors. Notice how we have put all the inductance into one wire for simplicity, but it makes no difference if you distribute it between the two wires. (Real lines also include resistance elements, but we will leave considering these until later.)

Let us make the initial supposition that $V$ and $I$ can vary along the wire and with time, i.e., $V = V(x, t)$ and $I = I(x, t)$. Now let’s look at what happens in a small “unit cell” of the diagram, of width $\Delta x$ — see Fig. 7. According to basic circuit theory, any voltage difference between points A and B must correspond to the back EMF across the inductor and hence to a

Figure 5: A simplistic view of a transmission line that does not agree with what we see in practice
Figure 6: How the two wires in a transmission line influence each other through the electric and magnetic fields that are created.

Figure 7: The transmission line redrawn as an “equivalent circuit” to include its inductance and capacitance. The dotted line represents the “unit cell” referred to in the text.

changing current through it. I.e.,

$$V(x + \Delta x, t) - V(x, t) = -\Delta L \frac{\partial I}{\partial t}.$$  \hspace{1cm} (14)

But \(\Delta L = L_0 \Delta x\) and so, taking the limit as \(\Delta x\) shrinks to zero, we can write

$$\frac{\partial V}{\partial x} = -L_0 \frac{\partial I}{\partial t}.$$  \hspace{1cm} (15)

What happens to the current as we move from \(x\) to \(x + \Delta x\)? Any extra current between points A and B must be caused by charge flowing off the
2 EXAMPLES OF WAVE PROPAGATION

capacitor. I.e.,

\[ I(x + \Delta x, t) - I(x, t) = \left( \frac{\partial Q}{\partial t} \right)_x = \frac{\partial}{\partial t} [ -\Delta C \ V(x) ] . \]  \hspace{1cm} (16)

When we shrink \( \Delta x \) down to zero again, this reduces to

\[ \frac{\partial I}{\partial x} = -C_0 \frac{\partial V}{\partial t} . \]  \hspace{1cm} (17)

(Notice the negative sign in front of \( C_0 \). This tells us that when an increased current flows through the inductor, charge flows off the capacitor and so \( V \) decreases.)

We can now use a little trick to obtain the wave equation very easily. Differentiate Eq. 15 with respect to \( x \) and Eq. 17 with respect to \( t \):

\[ \frac{\partial^2 V}{\partial x^2} = -L_0 \frac{\partial^2 I}{\partial x \partial t} \quad \text{and} \quad \frac{\partial^2 I}{\partial x \partial t} = -C_0 \frac{\partial^2 V}{\partial t^2} . \]  \hspace{1cm} (18)

This can be rearranged to give

\[ \frac{\partial^2 V}{\partial x^2} = L_0 C_0 \frac{\partial^2 V}{\partial t^2} . \]  \hspace{1cm} (19)

By comparing this with the Wave Equation, we see that \( c = 1/\sqrt{L_0 C_0} \).

2.5 Electromagnetic waves

The detailed theory of these waves is beyond the scope of the current course and will be described in 2EMs. However, we can summarise the situation as follows:

An EM wave consists of an oscillating “disturbance” in the electric and magnetic fields at successive points in space. A simple picture (Fig. 8) is often used to explain the oscillation, but it does not tell the whole story. Although
2 EXAMPLES OF WAVE PROPAGATION

Figure 8: Simple representation of the variation of $E_y$ and $H_z$ in an electromagnetic wave

the EM wave is described as a transverse wave, there is no suggestion that there is any “material” that actually oscillates — a hypothetical material called “aether” was originally thought to be the medium that “carried” EM waves, but Maxwell’s Equations show that such a material is not required. The situation is rather more abstract than this and is potentially difficult to visualise. Read Richard Feynman’s account of his efforts to appreciate what the fields “look like” (II-20-3).

The electric and magnetic fields for the wave shown in Fig. 8 are linked by the pair of equations below:

$$\frac{\partial E_y}{\partial x} = -\mu \frac{\partial H_z}{\partial t} \quad \text{and} \quad \frac{\partial H_z}{\partial x} = -\epsilon \frac{\partial E_y}{\partial t}. \tag{20}$$

These should look very familiar from the previous section and we can solve them to give

$$\frac{\partial^2 E_y}{\partial x^2} = \mu \epsilon \frac{\partial^2 E_y}{\partial t^2} \quad \text{and} \quad \frac{\partial^2 H_z}{\partial t^2} = \mu \epsilon \frac{\partial^2 H_z}{\partial t^2}. \tag{21}$$

Two for the price of one again. This is wave motion with $c = 1/\sqrt{\mu \epsilon}$. 
2.6 What do all the examples have in common?

2.6.1 The form of the wave equation

All the waves that we have so far looked at can be described by the same equation:

\[
\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2}.
\]

(22)

We will examine the significance of this more later, but for now, what you need to take on board is that whatever \( \psi \) represents, many key features of the wave motion will be the same. You do not have to relearn a whole different set of Physics for waves in transmission lines if you know how waves on a string behave.

2.6.2 Two key ingredients

The Physics describing all these cases has two key features:

- some kind of “elasticity” that provides a way of transferring the wave motion from one position in space to the next — for waves in a gas, for example, this corresponds to the inter-atomic forces involved in collisions, that cause one “element” of the gas to push on the next;

- an “inertia” term that keeps a given element going and “opposes” the push or pull of the elasticity term. In a gas or on a string, it corresponds to the density of the material.

How does this apply to the different wave equations studied so far?

We can clearly see that for the transmission line, \( 1/C_0 \) is identified with \( T \) and \( B_0 \), i.e., the elasticity term, whilst \( L_0 \) is like \( \rho \) and \( \rho_0 \) in the two
\[
\frac{\partial^2 y}{\partial x^2} = \frac{\rho \partial^2 y}{T \partial t^2} \quad \rightarrow \quad c = \sqrt{\frac{T}{\rho}}
\]

\[
\frac{\partial^2 \eta}{\partial x^2} = \frac{\rho_0 \partial^2 \eta}{B_0 \partial t^2} \quad \rightarrow \quad c = \sqrt{\frac{B_0}{\rho_0}}
\]  

(23)

\[
\frac{\partial^2 I}{\partial x^2} = \frac{L_0}{\frac{1}{c_0} \partial t^2} \quad \rightarrow \quad c = \sqrt{\frac{1/c_0}{L_0}}
\]

\[
\frac{\partial^2 H_z}{\partial x^2} = \frac{\mu}{\frac{1}{\epsilon} \partial t^2} \quad \rightarrow \quad c = \sqrt{\frac{1/\epsilon}{\mu}}
\]

Table 1: “Elasticity” and “inertia” terms for the examples studied in the course

mechanical cases. Notice that in all cases, the phase velocity of the wave, \( c \), is the ratio of the elasticity to the inertia. Notice that, in pursuing this analogy, we are effectively saying that empty space itself has certain properties, which is an unfamiliar concept.

### 2.6.3 Conjugate variables and analogies

Let’s explore another interesting feature, which, again, we see most easily in the mechanical examples. Consider our sound waves in a gas. The molecules are moved by a “forcing” term (the pressure) and exhibit some form of response (the displacement \( \eta \), or, alternatively, the velocity \( u_x \) — we could use either). We have two quantities of our system that both obey the wave equation:

\[
\frac{\partial^2 p}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} \quad \text{and} \quad \frac{\partial^2 u_x}{\partial t^2} = \frac{1}{c^2} \frac{\partial^2 u_x}{\partial t^2}. \]  

(24)
2 EXAMPLES OF WAVE PROPAGATION

String

\[ T_y = -T \frac{\partial y}{\partial x} \quad u_y = \frac{\partial y}{\partial t} \quad \frac{\partial T_y}{\partial x} = -\rho \frac{\partial u_y}{\partial t} \quad \frac{\partial u_y}{\partial x} = -\frac{1}{T} \frac{\partial T_y}{\partial t} \]

Sound

\[ p \quad u_x = \frac{\partial \eta_x}{\partial t} \quad \frac{\partial p}{\partial x} = -\rho_0 \frac{\partial u_x}{\partial t} \quad \frac{\partial u_x}{\partial x} = -\frac{1}{B_0} \frac{\partial p}{\partial t} \]

Trans. Lines

\[ V \quad I = \frac{\partial Q}{\partial t} \quad \frac{\partial V}{\partial x} = -L_0 \frac{\partial I}{\partial t} \quad \frac{\partial I}{\partial x} = -C_0 \frac{\partial V}{\partial t} \]

Electromagnetism

\[ E_y \quad H_z \quad \frac{\partial E_y}{\partial x} = -\mu \frac{\partial H_z}{\partial t} \quad \frac{\partial H_z}{\partial x} = -\varepsilon \frac{\partial E_y}{\partial t} \]

Table 2: “Conjugate” variables for the problems studied in the course

We shall call \( p \) and \( u_x \) conjugate variables. How are they related? From Eq. (11),

\[ \frac{\partial p}{\partial x} = -\rho_0 \frac{\partial^2 \eta}{\partial t^2} = -\rho_0 \frac{\partial u_x}{\partial t}. \] (25)

We will now develop a bit further the ideas of analogy between the various systems of equations? Look at the transmission line case again. We have two variables that are changing along the line, \( V \) and \( I \). We can again look on these as a forcing term \( V \) and a response \( I \). Looking back at Eq. (15), we see that it matches the form of the above equation for \( p \) and \( u_x \). The complete list of the conjugate variables and the relations between them is given in Table 2.

It is important to notice that each pair of equations carries the same information as the corresponding wave equation, since we can always use the trick of p. 10 to get one from the other. We will learn more about these conjugate variables in Section 5 when we talk about impedance.
3 Properties of the Wave Equation

3.1 Form and solutions of the equation in 1-D

We know that the equation \( \frac{\partial^2 \psi}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} \) (where \( \psi \) might be anything) corresponds to a propagating disturbance \( \psi(x, t) \), but we don’t know anything about its shape, only that it moves with speed \( c \). Suppose we consider a completely arbitrary shape at \( t = 0 \), as in Fig. 9 and denoted as \( \psi(x, 0) = f(-x) \). (The \(-\) sign here is purely a convention.) As long as the shape does not change when it propagates, the solution at an arbitrary time \( t \) is

\[
\psi(x, t) = f(ct - x) \quad \text{or} \quad f(\xi) \quad \text{where} \quad \xi = ct - x
\]  

(27)

This is the most general form of the solution to the 1-D wave equation.

Exercise: Use the chain rule to do a partial differentiation of \( \xi \) w.r.t. \( x \) and \( t \) and hence show that \( f(\xi) \) satisfies the Wave Equation. You should notice that we can use either a \(+\) or a \( -\) sign in front of the \( x \). What do these represent?

Notice that this functional form is very general — any shape will propagate. In order to get any further, we need a particular solution of the partial DE and in order to get this, we need some boundary conditions. Typical conditions are \( \psi(x, 0) \) for all \( x \) or \( \psi(0, t) \) for all \( t \). To give a concrete example, if I want to look at voltage waves down a transmission line, I need to know either what the voltage is everywhere at some particular time, or I need to know what is happening at the signal generator for all different times.
3 PROPERTIES OF THE WAVE EQUATION

3.2 Linearity and Fourier analysis

A key feature of the Wave Equation is that if \( f(ct - x) \) and \( g(ct - x) \) are both solutions then so is \( h(x,t) = f(ct - x) + g(ct - x) \). This property is known as linearity and, though it may seem obvious, there are many partial differential equations for which it isn’t true; we will be looking at some of these in Section 3.3. Linearity has two important consequences:

- Although waves interfere constructively or destructively, the superposition of amplitudes is all that happens. The presence of \( f \) does not affect how wave \( g \) behaves and vice versa. If the two waves are travelling in opposite directions, they pass right through each other.

- Superposition allows us to build up a general waveform with a complicated shape by adding together small amounts of lots of waves with simpler shapes.

In Maths, one learns that any periodic function, with period \( L \) can be built up from a set of exponentials as follows:

\[
f(x) = \sum_{j=-\infty}^{\infty} \alpha_j e^{-ik_j x} \quad \text{where} \quad k_j = j \cdot \frac{2\pi}{L}.
\]  

(28)
This is known as a *Fourier series*. Although we will not prove this relation here, you should make yourself familiar with the method of Fourier series.

**Exercise:** Show that the Fourier series representation of the square-wave

\[
f(x) = \begin{cases} 
0 & -\lambda/2 \leq x < 0 \\
1 & 0 \leq x < \lambda/2
\end{cases}
\]  

(29)

can be written using the equation above with the following coefficients \(\alpha_j\):

\[
\alpha_0 = \frac{1}{2} \quad \text{and} \quad \alpha_{n \neq 0} = \begin{cases} 
\frac{1}{\pi n} & n \text{ odd} \\
0 & n \text{ even}
\end{cases}
\]  

(30)

Even more generally, any function of \(x\) can be built up from an integral like this:

\[
f(x) = \int_{-\infty}^{\infty} \alpha(k)e^{-ikx} \, dk.
\]

(31)

This is a *Fourier integral*.

The whole subject of how to build up a particular wave shape from a set of exponentials and what the \(\alpha(k)\) values are is known as *Fourier analysis*. We won’t go into much detail here, although we will meet the subject again when we deal with Fraunhofer diffraction in Section 6. The important thing to notice here is that it works because the Wave Equation is linear. Eq. (31) above is for the case of a static waveform, but we can also include an \(\omega t\) dependence to describe travelling waves.

The basic building blocks of Fourier analysis are the individual waves of form \(e^{-ikx}\) or \(e^{i(\omega t - kx)}\). These are often called *plane wave* solutions of the Wave Equation, for reasons that we will see later, and are very important. Notice that, in almost all of the problems that you will come across (with the important exception of Quantum Mechanics) it is the *real* part of this function...
Figure 10: Non-linear wave propagation in ultrasound: (a) original waveform; (b) modification in wave shape as it propagates; (c) multi-valued solution not possible in practice; (d) final form of the wave, showing shock-front behaviour that represents the quantity that is actually observed. (The imaginary part helps us keep track of the phase of the oscillation, but we will not discuss this further here.)

3.3 Non-linear wave motion

In this section we take a brief overview of what happens in more complicated cases when the Wave Equation for a given problem is not linear. We will use two examples, from the fields of ultrasound and tidal waves, and will encounter some rather more unusual results. The mathematics of non-linear wave motion are generally speaking more involved than that we have so far been dealing with, so no details will be presented here. See Pain Ch. 15 for further information.

3.3.1 Non-linear effects in ultrasound

When we derived the equations for the propagation of sound waves, we took a number of shortcuts and made a number of approximations. If one does a
more careful derivation, one obtains an equation of the form

\[ \frac{\partial^2 \eta}{\partial x^2} = \frac{1}{c_0^2 (1 + s)^{\gamma + 1}} \frac{\partial^2 \eta}{\partial t^2}, \]

(32)

where \( s \) is the so-called \textit{condensation} of the gas, i.e., \( \rho/\rho_0 \), and \( \gamma \) is a constant. Physically, this has a very interesting meaning. The equation implies that the speed of the wave motion is not constant, but \textit{varies with} \( s \); the elements of the wave where the condensation (and hence density) are high will have a wave speed that is \textit{higher} than those parts of the wave where the density is lower. In other words, the \textit{compressions travel faster than the rarefactions}. This is a very intriguing behaviour and it is shown schematically in Fig. 10.

We see that as it propagates, a wave that starts off as sinusoidal (a), gradually changes shape (b), as the crests move forward relative to the troughs. The shock fronts in (d) can be used beneficially in \textit{lithotripsy} to shatter kidney stones.

### 3.3.2 Tidal Bores

An example of a very interesting phenomenon that occurs in a number of tidal rivers, one of the most famous being in the River Severn, which divides England from South Wales, is a tidal bore. Suppose we plot the height of the water at a particular point as a function of tide. At the edge of the estuary, near the open sea, it has a sinusoidal form. If one proceeds a little further up-river, then we have a very similar form, but \textit{out of phase} with the first oscillation. The further up-river one goes, the greater the phase lag. Now this corresponds exactly with what we might expect for a wave motion. It turns out that we can regard the tide in the river as a wave motion that moves up-river with a speed of several tens of kilometres per hour.

However, life is not quite this simple. It turns out that if we plot the \textit{shape}
of the wave more carefully as we go further up-river, it starts to change. Just as in the ultrasound case, the surface of the wave seems to “bunch up” at the front and a noticeable shock front starts to form. At certain points along the river, this shock front is a very steep wall up to 2 m high and at some times of the year one can surf several hundred yards, moving along with the shock front, which in this particular case is called a tidal bore.

4 More than one dimension

4.1 Introduction

The first example of waves that most people ever see is a stone thrown into a pond. This is an example of a 2-D transverse wave motion — the waves can move anywhere on the 2-D surface of the pond and the water oscillates (approximately) in the vertical direction, transverse to the direction of propagation. Surprisingly, it turns out that the Physics of water waves is actually quite complicated and a much simpler 2-D problem is the vibration of a drum skin (Appendix A).

4.2 What sort of wave motion can we have?

It soon becomes clear that, in 2D, there is a greater range of wave motion possible. Waves can move in any direction, not just along $x$ or $y$, and we can have circular waves, too. However, all these possibilities are valid solutions of the 2-D Wave Equation Eq. (79).

One very important solution (as in 1D) is the plane wave $z = e^{i(\omega t - [k_x x + k_y y])}$.

[Exercise: Verify that this is indeed a solution of Eq. (79).]
Notice that if a general position in the \((x, y)\) plane is \(\mathbf{r} = \begin{pmatrix} x \\ y \end{pmatrix}\) then \(k_x x + k_y y = \mathbf{k} \cdot \mathbf{r}\), where \(\mathbf{k} = \begin{pmatrix} k_x \\ k_y \end{pmatrix}\). This means that we can write our plane wave solution as

\[
  z = e^{i(\omega t - \mathbf{k} \cdot \mathbf{r})}
\]  

(33)

\(\mathbf{k}\) is called the wave vector. If we try to draw what the wave motion looks like, then we see that the plane wave has a sinusoidal form, but “stretched out” over the second dimension like a piece of corrugated iron. There is a set of regularly spaced wave crests \(\lambda = 2\pi/|\mathbf{k}|\) apart and these move in the direction given by the unit vector \(\hat{\mathbf{k}}\) at a speed \(c = \omega/|\mathbf{k}|\). Notice, once again, that the physical observable is the real part of this complex exponential.

Whilst we won’t go into any detailed proofs, you should be able to see that we can make up any shape of surface by a suitable combination of plane wave solutions. Thus,

\[
  z(x, y, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \alpha(k_x, k_y) e^{i(\omega t - [k_x x + k_y y])} \, dk_x dk_y
\]  

(34)

represents a general form of solution. However, the expression for \(\alpha(k_x, k_y)\) can get quite complicated.

### 4.3 3-D solutions

Comparing Eqs. (22) and (79), a logical extrapolation would be to assume that we can represent waves in 3D of some property \(\psi(x, y, z)\) by an equation of the form

\[
  \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2}.
\]

(35)
This is indeed the case and, just as above, we can find plane wave solutions of the form

$$\psi(x, y, z, t) = \psi(r, t) = e^{i(\omega t - k \cdot r)}$$

with $k \cdot r = (k_x, k_y, k_z) \cdot (x, y, z)$. (36)

Again, the direction of propagation is $\hat{k}$ and the wavelength is $\lambda = 2\pi/(k_x + k_y + k_z)^{1/2}$. Mathematically, we have a simple step up in the dimensionality of the problem, but what do the solutions “look like”?

At any given time, all points where $k \cdot r = \text{const.}$ have the same phase. This means that all points in a given plane perpendicular to $k$ have the same value of $k \cdot r$. We can thus envisage our plane wave motion as a series of planes of constant $\psi$ that are advancing at speed $c$.

Now we need to think a little more about what $\psi$ actually is — the above idea is a bit abstract and mathematical. $\psi$ might, for example, be the particle displacement $\eta$ along the direction of longitudinal wave propagation. (To think about: Why can we not have transverse waves in the same way as for 1D and 2D?) $\psi$ represent pressure, temperature or electric field, in fact any property that varies spatially. In general, it is very difficult to visualise what the 3-D waves look like. Read Sec. II-20-3 of Feynman to see a Nobel Prize Winner’s attempts. Again, we can use the principle of superposition to make 3-D waves using Fourier analysis — you should be able to write down the formulae yourself — and this means that any of the complicated patterns of waves we find in Nature (light, sound, etc.) can be split into many (possibly an infinite number of) “monochromatic” plane waves. You might say that our eye acts as a natural Fourier analyser, by splitting up a 3-D wave of $\mathbf{E}$ and $\mathbf{H}$ vectors into its component colours.
4.4 Non-cartesian co-ordinate systems

The plane wave representation is particularly convenient when we have a problem with “lots of straight sides” (e.g., waveguides with rectangular cross-sections). However, many of the most frequently encountered problems involve point sources emitting waves in all directions. The solution to the 3-D Wave Equation that describes spherical waves is

\[ \psi(r, t) = \frac{A}{r} e^{i(\omega t - kr)}. \]  

(37)

It shouldn’t take you long to convince yourself that the algebra required to verify that this is indeed a solution if Eq. (35) is very tedious. (The adventurous among you could experiment with a symbolic algebra package — xmaple is available on the UNIX systems — to perform the relevant differentiations.) It turns out that working in cartesian co-ordinates (i.e., \(x, y\) and \(z\)) is a bad thing to do when you have a problem involving spherical symmetry. Instead, we pick a different co-ordinate system involving \(r\). The spherical polar co-ordinate system describes every point \(P\) in terms of \((r, \theta, \phi)\) instead of \((x, y, z)\) — see Boas or another Maths textbook for further details.

In this particular case, the problem doesn’t even depend on \(\theta\) and \(\phi\). Given that this is the case, our 3-D Wave Equation can itself be transformed into spherical polars leading to

\[ \frac{\partial^2 \psi}{\partial r^2} + \frac{2}{r} \frac{\partial \psi}{\partial r} = \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2}. \]  

(38)

Problem 4 involves verifying that our function \((A/r^2)e^{i(\omega t - kr)}\) is a solution to (38).

It is important to note that, although this equation looks very different from our previous statement of the 3-D Wave Equation, Eq. (35), they are
String \( Z = T_y/u_y \) \quad \left( = \frac{-T \partial y/\partial x}{\partial y/\partial t} \right)

Sound \( Z = p/u_x \) \quad \left( = \frac{p}{\partial \eta_x/\partial t} \right) \tag{40}

Transmission Lines \( Z = V/I \) \quad \left( = \frac{V}{\partial Q/\partial t} \right)

Electromagnetism \( Z = E_y/H_z \)

Table 3: Definition of impedance for the problems studied in the course

in fact just different representations of the same thing. We can highlight this by using a common notation in mathematical Physics, the \( \nabla^2 \) operator, also called the Laplacian. A very compact way of writing the Wave Equation is

\[
\nabla^2 \psi = \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2}.
\] \tag{39}

You can regard the Laplacian as “embodying” the mathematical properties of the “space” in which the waves propagate. It is then up to us to choose a co-ordinate system that best matches our problem and for each there is a corresponding representation of the Laplacian. The three most common in Physics are cartesian, cylindrical polars and spherical polars. For the exam, you don’t need to know all the representations, but you do need to understand the general idea. Notice also that in problems that are analysed in spherical polars, we tend to build up complex solutions not from plane waves, but from other basis functions, called spherical harmonics.
5 Impedance

5.1 Introduction and definition

You should be familiar from Level 1 Electronics with the idea of impedance in an electric circuit. In this section, we will see that impedance is in fact a much more general concept that helps explain and unify a wide variety of phenomena associated with waves and vibrations. The usefulness of impedance as a concept is a consequence solely of the mathematical form of the Wave Equation. Any situation that can be expressed in this form will have an associated impedance.

In the study of electricity, \( Z = V/I \). It is a relationship between a “forcing” term \( V \) and a “response” \( I \). \( V \) is what pushes current \( I \) and \( Z \) is the constant of proportionality that tells us “how much \( I \) we get for a given \( V \)”.

I will now assert that we can do this generally. We define, for each problem studied, a quantity called the impedance, where

\[
Z = \frac{\text{Forcing term}}{\text{Response term}}.
\]  

(41)

By learning what the properties of impedance are in one field, e.g., Electromagnetism, we can figure out what happens in any other field simply by swapping the names of the relevant variables. Look back at Sec. 2.6.3. \( V \) and \( I \) are what I called conjugate variables and the table below defines impedances for the other situations in terms of their appropriate conjugate variables.
5.2 Specific and characteristic impedance

The impedance associated with a propagating plane wave is particularly important and often written $Z_0$. Suppose that, in our transmission line example, $V = V_0 e^{i(\omega t - kx)}$. Looking back at Table 2, we see that

$$\frac{\partial V}{\partial x} = -L_0 \frac{\partial I}{\partial t}.$$  \hspace{1cm} (42)

Using the useful trick that $\partial/\partial x \to -ik$ and $\partial/\partial t \to i\omega$ for a plane wave, the equation becomes

$$-ikV = -L_0i\omega I$$

$$\iff Z_0 = \frac{V}{I} = \left(\frac{\omega}{k}\right) L_0 = L_0 c = \sqrt{\frac{L_0}{C_0}}.$$ \hspace{1cm} (43)

Notice that this result contains only information concerning the characteristic properties of the transmission line itself. It is thus called the line’s characteristic impedance. Notice that it is the product of the wave speed and the “inertia” term of Table 1. By the usual method of just changing the variable names, the corresponding results for the other systems are:

- string $Z_0 = \rho c = \sqrt{\rho T}$;
- sound in gas $Z_0 = \rho_0 c = \sqrt{\rho_0 B_0}$;
- electromagnetism $Z_0 = \mu c = \sqrt{\mu/\epsilon}$.

In the last example, the medium that EM waves are propagating through has a characteristic value of $Z_0$ and so it seems that free space itself has an impedance, which is rather a strange concept. In fact, this impedance is $Z_0 = \sqrt{\mu_0/\epsilon_0} \approx 377 \Omega$.

The result above applies only for forward travelling plane waves (in the 1-D case). For all other types of wave the relevant term is the specific impedance, which is given by $V(r, t)/I(r, t)$ and does not (necessarily) reduce to an expression involving just the characteristics of the propagating medium. The result obtained will be specific to the shape of wave.
Let us consider again our example of a spherical acoustic wave. Notice from
the formula above that the specific impedance depends on position. In this
particular case, the problem is spherically symmetric and so $Z$ will depend
only on $r$. Problem 4 asks you to show that

$$u_r = \left(1 - \frac{i}{kr}\right) \frac{p}{\rho c r}.$$  \hspace{1cm} (44)

This result implies that the impedance is

$$Z = Z(kr) = \rho c \frac{kr}{1 + (kr)^2} (kr + i).$$  \hspace{1cm} (45)

and the important features to note are:

(a) The impedance has both a real and an imaginary part. The real part
tells us that energy is transported outwards by the wave, while the
existence of an imaginary component tells us that the particles vibrate
slightly out of phase with the pressure variations.

(b) $Z$ varies with $k$ and $r$ such that neither appears by itself, but always
the product $kr$. This is a common feature of wave problems (e.g., also
diffraction) and tells us that the problem scales with wavelength.

(c) When the product $kr \gg 1$, the imaginary component tends to zero,
leaving $Z(kr) \to \rho c$. This means that at distances that are sufficiently
large compared with the wavelength, the spherical wave appears, lo-
cally, just like a plane wave, which is what you might expect.

This example and one other are treated in further depth in Appendices B.4
and B.5.
5.3 Transmission at boundaries

5.3.1 Introduction

When an incident wave travelling through a medium with characteristic acoustic impedance $Z_1$ hits a boundary with a second material with impedance $Z_2$, then partial reflection occurs. A reflected wave and a transmitted wave appear and the amplitudes of these are related to the difference in characteristic impedance. This is an important general phenomenon and what you learn here will be applicable to phenomena as diverse as the manufacture of anti-reflective coatings for spectacles to the design of ultrasound probes for use in the clinic. The reason for this phenomenon can be shown by a relatively simple solution of the Wave Equation, with boundary conditions.

5.3.2 The 1-D case — normal incidence

This solution of the 1-D wave problem is appropriate for waves arriving at the junction between two strings made of different materials, for ultrasound or light waves incident perpendicular to a boundary and also for signals travelling down a transmission line.
Consider Fig. 11, which shows what happens when a plane wave hits an interface between two media with different values of $Z$. Appendix B.1 shows that the reflection and transmission coefficients are given by

\[ R = \frac{Z_2 - Z_1}{Z_2 + Z_1}, \quad T = \frac{2Z_2}{Z_2 + Z_1}. \] (46)

Thus, the transmitted and reflected waves have amplitudes which are dependent on the properties of the media on either side of the interface. Note that, if $Z_1 > Z_2$, the reflection coefficient is negative. This implies a change of phase of $180^\circ$ on reflection at the boundary.

This result is particularly relevant to the experiment “Transmission Lines” in the Level 2 Electromagnetism lab, which will help you see practically how the formula is applied.

Three notes of caution: First, Eq. (46) refers to the reflection coefficients for wave amplitude. If you are doing a problem that involves power (intensity), you must remember that $W = \frac{p^2}{Z}$ for ultrasound and similarly for the other problems. When you calculate the intensity transmission coefficient in medium 2, be particularly careful. Second, Eq. (46) applies to only one of the conjugate variables ($p$ in the ultrasound situation, $V$ for transmission lines, etc.). If you want to look at the other variable, the sign of $R_0$ is reversed, i.e., use $(Z_1 - Z_2)$ in the numerator. See the Table on p. 548 of Pain for further clarification. Finally, recall that in the derivation of these results, we used plane wave solutions of the Wave Equation. Such “infinite plane waves” are an ideal mathematical case and do not occur in Nature. Hence, the results here are not strictly correct for any of our real-life problems. However, they will often be adequate for our purposes.
5.3.3 Non-normal incidence

The derivation above can be extended to 3D relatively easily — see Fig. 12. However, the algebra extends to several pages and is not examinable. Details are included in Appendix B.2 for those interested. The important feature to notice is that the 3-D solution leads to some new results, which we can group together as the Laws of Reflection and Refraction:

(i) The wave vectors of the incident wave, the transmitted wave and the reflected wave all lie in the same plane.

(ii) The angle of the incident wave to the normal of the boundary plane is the same as the angle of the reflected wave. (Law of Reflection)

(iii) \( \frac{\sin i}{\sin t} = \frac{c_1}{c_2} \) for sound waves (Snell’s Law).

(iv) The amplitudes of the incident and reflected waves are given by

\[
R_0 = \frac{Z_2 \cos i - Z_1 \cos t}{Z_2 \cos i + Z_1 \cos t} I_0 \quad \text{and} \quad T_0 = \frac{2Z_2 \cos i}{Z_2 \cos i + Z_1 \cos t} I_0 \quad (47)
\]
5.3.4 Ultrasound imaging

Ultrasound imaging, either in a medical context or industrially, is a perfect example of the application of the above piece of Physics. Imaging is based on the same principle as sonar, used for the detection of objects underwater, and the echo-location method used by bats.

Pulses of ultrasound are emitted from a transducer. They propagate through the medium until they come to a boundary. In Medicine, this will be the boundary between two different tissues — see Fig 13 — whilst in industrial ultrasound, it might be the boundary of a flaw in the item being tested. At this boundary, there is a sudden change in $Z$, which causes reflection. The reflected pulse returns to the transducer and a signal is registered. By looking at the time between the outgoing pulse and the receipt of the reflected
pulse, the distance to the boundary can be obtained using the simple relation 
\[ d = \frac{ct}{2}. \]

The formulae above explain quantitatively what governs the \textit{contrast} in ultrasound images. Fig. 14 shows an ultrasound image of a 19-week old foetus in the womb. Notice that the regions which show up brightest correspond to bony features. This is because the difference between \( Z_{\text{bone}} \) and \( Z_{\text{soft tissue}} \) is much greater than the difference between the \( Z \) values of two different soft tissues. Hence \( |R_0| \) is much greater at the interfaces between bone and soft tissue than elsewhere.

Reflection at boundaries can often be a significant problem. In diagnostic ultrasound, the impedance difference between the probe material and air is very large. This means that the reflection coefficient at the boundary is very high (\( p_r/p_i \approx -0.9997 \)). Hence, if there is the slightest trace of air when the probe is pressed against the body, virtually no ultrasound gets through and imaging is impossible. To combat this problem, a thin layer of gel is smeared onto the body first. This has the effect of excluding any air and \textit{couples} the probe to the body. (Even so, a lot of the signal is still reflected at the interface (\( p_r/p_i \approx 0.86 \)) because \( Z_{\text{transducer}} \gg Z_{\text{tissue}} \).

\subsection*{5.3.5 Impedance matching}

There exists a technique called impedance matching that can be used as a way around the problem. Again, it relies on some relatively simple Physics. Consider Fig. 15, which shows the original interface in (a) and a modification (b), whereby an extra layer is inserted. If this layer has exactly the right thickness, i.e., \( \lambda/4 \), where \( \lambda \) is the wavelength in the “matching layer” and exactly the right impedance \( Z_3 = \sqrt{Z_1 Z_2} \) then, almost magically, all
the incident radiation passes through, without being reflected. The way in which we derive this result is, again, simply to solve the Wave Equation with appropriate boundary conditions, as in Sec. 5.3.2. However, since the algebra runs to a couple of pages, we will not undertake it here. For those interested, see Appendix B.3 or Pain p. 127ff.

Impedance matching has many important uses in everyday life:

- design of ultrasound probes — If we coat the surface of the probe with a layer of impedance $\sqrt{Z_{\text{probe}}Z_{\text{tissue}}}$, we can potentially improve the transmission of ultrasound into the body.

- lens manufacture — Light is reflected at the surface of a lens and this can significantly reduce the amount of light reaching the film. Coating the surface of the lens with a very thin layer of a dielectric substance for which $Z = \sqrt{Z_{\text{air}}Z_{\text{glass}}}$ makes the lens more efficient.
Figure 15: Illustration of impedance matching in a diagnostic ultrasound probe

- electronics — The power transfer from a generator to its load is maximum when the input impedance of the load is the same as the output impedance of the generator. A practical example is in any hi-fi system, where the loudspeaker impedance must match the output impedance of the amplifier. In high frequency applications, an electrical transmission line is matched to its load by inserting a cable of length $\lambda/4$ with appropriate impedance.

It is important, however, to realise a key limitation of impedance matching. Because of the requirement that the matching layer be exactly $\lambda/4$, we only get 100% transmission for a single wavelength of radiation. This explains why, for example, the surfaces of lenses often look purple. The matching is not working for short wavelength radiation. It also proves a disadvantage in ultrasonics when we wish to transmit pulses, rather than continuous waves, across the boundary. We have seen how a pulse waveform can be regarded as being made up of a Fourier superposition of many wavelengths. If such a pulses hits a boundary, then the different wavelengths composing it will be
transmitted to different degrees (i.e., \( R_0 = R_0(\lambda) \)). The pulse that emerges at the other side of the boundary will have had its shape changed, as some of its components have been filtered out.
6 Diffraction

6.1 Making Huygen’s wavelets “quantitative”

We are all used to the simple concept of Huygen’s wavelets, as shown pictorially in Fig. 16. But how can we use this mathematically and without having to draw all those fiddly diagrams? Fresnel proposed that we could represent all the points on the wavefront by individual sources of spherical waves. In principle, each spherical wave would give rise to a contribution to the total wavefront $\Psi$ given by

$$d\Psi \propto e^{i(\omega t - kr)} \frac{1}{r}$$

(48)

where $r$ is the distance from the point on the original wavefront (i.e., the origin of the secondary wavelet) to the point where we are making a measurement. Notice the use of $\Psi$ to describe our wave. $\Psi$ represents a general
property that exhibits wave motion — our results will be applicable to any
type of wave motion in more than one dimension. Note also that, in the fol-
lowing explanation, we will ignore all constants of proportionality and comp-
lating factors that are not directly relevant to the shape of the diffraction
pattern.

Consider the situation shown in Fig. 17. Waves of form $\Psi(x, t) = e^{i(\omega t - kx)}$
are incident from the left. These hit an obstacle at $x = 0$ and we look at the
Huygen’s wavelet emitted from a typical point $Q$ at height $y$ compared with
the origin $O$.

We are interested in evaluating the contribution $d\Psi$ of this wavelet at
point $P$. First, we need to describe the obstacle. This is done by an aperture
function, $a(y)$. In the simplest case, $a(y)$ is 1 where the wave can get through
and 0 where it is blocked. In more complex examples, we can vary the
amplitude of $a(y)$ to represent partial transmission or introduce an arbitrary
phase.

$$d\Psi(\text{at } P \text{ due to } Q) = d\Psi(X, Y) = a(y) \, dy \, \frac{e^{i(\omega t - kr)}}{r}$$

where $r = (X^2 + (Y - y)^2)^{1/2}$ is the distance between points $P$ and $Q$. 
The overall value of \( \Psi \) is the result of summing all the individual contributions \( d\Psi \), i.e., doing an integration. It is difficult to go further without approximations. First, replace \( r \) as follows:

\[
    r \approx \sqrt{X^2 + Y^2 - 2yY} = R \left[ 1 - \frac{2yY}{R^2} \right]^{1/2},
\]

(50)

where \( R \) is the distance from \( O \) to \( P \), i.e., \((X^2 + Y^2)^{1/2}\). Now use a Taylor expansion:

\[
    r \approx R \left[ 1 - \frac{yY}{R^2} + \ldots \right] \\
    \approx R - \frac{yY}{R} \\
    \approx R - y \sin \theta,
\]

(51)

where \( \theta \) is the angle between the straight-ahead direction and the line \( OP \). Hence, Eq. (49) becomes

\[
    d\Psi(X,Y) = a(y) \, dy \, e^{ik(R - y \sin \theta)}.
\]

(52)

We then argue that, whilst the extra phase given by the \( y \sin \theta \) term is important, the extra amplitude is not. (This is true for \( y \ll R \)). Finally, the expression for the total signal at \( P \) is given by

\[
    \Psi(X,Y) = \frac{e^{i(\omega t - kR)}}{R} \int_{-\infty}^{\infty} a(y) \, e^{iky \sin \theta} \, dy
\]

(53)

This important formula is known as the *Fraunhofer* integral and plays the primary role in understanding diffraction patterns at a large distance from the obstacle. Notice that the waveform depends on the angle \( \theta \) rather than on the particular values of \( X \) and \( Y \) and also the pattern scales with \( k \) and hence \( \lambda \). We will make a further notational change and consider only the
latter term in the equation, which represents the *envelope* of the diffraction pattern. (The first part of the formula represents simply the oscillatory pattern of a spherical wave, whose amplitude decays with 1/R.) We will call the envelope function $\psi$, where

$$\psi(q) = \int_{-\infty}^{\infty} a(y) e^{iqy} dy .$$  \hfill (54)

Introducing $q = k\sin\theta$ is a minor but important change, the purpose of which is to make the equation look like a Fourier transform — see Sec. 3.2.

### 6.2 The single slit ↔ plane piston transducer

Because of the way we have couched our explanation, in terms of a general variable $\psi$, we can now kill two birds with one stone again. Consider single-slit optics experiment and compare this with a simple rectangular ultrasound transducer (often called a plane piston transducer). Both correspond to a “portion of wavefront” that can be described by an aperture function

$$a(y) = \begin{cases} 1 & -b/2 < y < b/2 \\ 0 & \text{otherwise} \end{cases}$$  \hfill (55)

The Fraunhofer integral now becomes

$$\psi(q) = \int_{-b/2}^{b/2} 1 \cdot e^{iqy} dy$$

$$= \left[ \frac{e^{iqy}}{iq} \right]_{-b/2}^{b/2}$$

$$= b \text{sinc} \frac{b}{2}q .$$  \hfill (56)

Notice that what we end up with is a function of $q = k\sin\theta$ and that the particular function in question is the sinc function (short for “cardinal sine”),
Figure 18: Possible methods of display for the function $\psi$. (a) is a typical transducer beam profile. (b) is a plot of intensity ($\varpropto |\psi^2| \cdot \sin \theta$). (c) is the equivalent pattern of fringes as might be observed in an optics experiment. These diagrams were simulated for the case of $b = 6\lambda/\pi \approx 2\lambda$. Note that it is difficult to see the sidelobes in the greyscale image, as they are a weak feature.

where $\text{sinc } x = \sin x/x$. The result can be displayed in one of several ways — see Fig.18.

6.3 Young’s Slits

This is again a commonly encountered problem. It might also be applied to two ultrasound transducers close to one another or two radio transmitter
masts close to each other. In this case the aperture function is

$$a(y) = \begin{cases} 
1 & -b < y < -a \quad \text{and} \quad a < y < b \\
0 & \text{otherwise}
\end{cases}.$$  \hfill (57)

Since the Fourier transform is linear, we can regard this as the sum of two terms:

$$\psi(q) = \int_{-b}^{b} 1 \cdot e^{iqy} \, dy - \int_{-a}^{a} 1 \cdot e^{iqy} \, dy = bsinc\frac{b}{2q} - asinc\frac{a}{2q}.$$  \hfill (58)

Note that when the slits are infinitesimally thin (line sources), we can formulate the problem in a different way:

$$a(y) = \delta(y + b) + \delta(y - b),$$  \hfill (59)

which leads to the Fraunhofer integral

$$\psi(q) = \int_{-\infty}^{\infty} \delta(y + b) e^{iqy} \, dy + \int_{-\infty}^{\infty} \delta(y - b) e^{iqy} \, dy$$  \hfill (60)

Remembering that the integral of the delta function has the effect of sampling the accompanying function at the location of the delta function, we get

$$\psi(q) = e^{iqb} + e^{-iqb} = 2 \cos qb.$$  \hfill (61)

This result is very famous and describes the pattern of fringes that is seen in the classic optics experiment Young’s slits.

The results above are ones that are often treated by more elementary methods. The real power of the Fraunhofer integral comes from the fact that it allows us to work out the result for any arbitrary shape of aperture. It can be extended to allow shaped 2-D apertures, by making the step $a(y) \rightarrow a(x, y)$
and evaluating a 2-D Fraunhofer integral. One can also account for cases (particularly in the design of complex ultrasonic transducers) where waves emanating from different parts of the aperture oscillate with different phases. By allowing $a(x,y)$ to be a complex function, we can introduce these phase factors in a very straightforward fashion. This can be put to good use in medical diagnosis, where the beam of ultrasound produced by a linear array transducer can be scanned to and fro across a sample by simply changing the phases of signals reaching different transducers in the array.

### 6.4 Fresnel diffraction

Having enthused about the merits of using the Fraunhofer integral above, we now have to admit that there is still a lot it doesn’t tell us. We know only about what is happening at large distances (i.e., $d \gg \lambda$) from the aperture. This is because of the approximations made in Eqs. (50) and (51). Closer to the aperture — “closer”, here, being defined by where the terms that are quadratic in $y$ are significant — things become more challenging. There tends to be no simple analytical solution to the equations. Under these circumstances, we have Fresnel diffraction. An excellent description of this whole area is to be found in Chapter X of *Geometrical and Physical Optics* by R. S. Longhurst, 3rd Ed., Longman. It explains from first principles (i.e., the Wave Equation) where the whole concept of Huygens wavelets comes from and gives the “complete” diffraction theory, of which you have had just a small taste here.
7 Attenuation by scattering and absorption

7.1 Introduction

Absorption and scattering are the phenomena by which organised vibrations (i.e., wave motion) are transformed into disorganised random motion. The mechanisms for this are many and varied. There will not be time in this course to discuss them in detail, so this section will be much more of an overview. An interesting discussion of the various processes involved for electromagnetic radiation can be found in Longhurst, “Geometrical and Physical Optics”. As far as ultrasound goes, many of the effects have never even been investigated fully for substances as complex as human tissues and descriptions are generally to be found in the primary scientific literature or review articles, rather than elementary textbooks.

7.2 Scattering

At its simplest and on a large lengthscale (i.e., $\gg \lambda$), we can regard scattering simply as a series of multiple reflections. A good example is a beam of ultrasound incident on lung tissue. Lungs consist of a large spongy mass of “tubes”, branching off each other at an ever decreasing scale. They have evolved to have a large surface area for efficient oxygen exchange, but this leads to an enormous number of boundaries at which ultrasound is reflected in random directions. If we consider the “straight on” beam, it is highly attenuated, but there is no overall reflected beam; the light is scattered in all directions.

When the “scattering centres” are much smaller, then a phenomenon called Rayleigh scattering occurs. We shall investigate this for electromagnetic
waves. Feynman I-32 gives a good account, but the basic idea is as follows:
Light waves incident on an atom cause the positive and negative charges to
oscillate in opposite directions, as in Fig. 19. This creates what is known as
an oscillating dipole. Using Electromagnetism theory that is slightly more
advanced than that done at Level 2, it can be shown that this is the source or
electromagnetic radiation and that this radiation is emitted in all directions.
I.e., the incident beam is scattered.

Rayleigh scattering helps to explain a number of natural phenomena con-
cerned with sunlight. Why is the sky blue and why are sunsets orange? It
turns out that the degree to which light is Rayleigh scattered is proportional
to $f^4$, i.e., the higher the frequency, the more light is scattered. So, as the
sun passes through the Earth’s atmosphere lots of blue light (high $f$) is scat-
tered, while the red end of the spectrum passes through almost undisturbed.
The light that reaches us from parts of the sky not in the direct path of the
sunlight is the result of multiple scatter and is thus blue. In the evening, the
sunlight that we see has passed through a large length of atmosphere and
almost all the blue light has been scattered *out* of the beam. Thus, what reaches us appears red.

Similar arguments help us understand why infra-red cameras can help us see through mist better (lower wavelength IR is scattered less — although not for thick fog, where we have particles much bigger than $\lambda$ and it is not Rayleigh scattering) and why using a UV filter on your camera will give you better results in taking pictures in hazy conditions. In ultrasound imaging, it is well known that the higher the frequency of ultrasound used, the less well it penetrates tissue and this is a major consideration in the design of equipment.

### 7.3 Absorption

As mentioned above, the mechanisms of absorption are complex. For waves in a string, wave energy is absorbed and turned into heat by the continual stretching and contraction of the string and this in turn can be related to the microscopic forces between molecules — try stretching and releasing a rubber band many, many times and it will heat up. Ultrasound waves in the human body lose energy through viscosity, through coupling to molecular vibrations and by causing chemical changes in tissues. Energy losses in transmission lines can be explained by Joule heating (i.e., $I^2R$), but this isn’t really a full explanation and to really understand this, one needs to think on a microscopic level about electrons drifting along under the influence of $V$ and colliding with atoms. Finally, in the case of electromagnetic waves, one needs to think about “damping processes” that occur as the dipole of the previous section oscillates.
7.4 Attenuation in the Wave Equation

Let us quickly remind ourselves that Quantum Theory tells us that light can be regarded both as waves and particles. Let us think about how scattering might be viewed in a particle-based picture. Suppose particles (photons), travelling in the $+x$-direction are incident at random positions on a homogeneous material. It is plausible to suppose that the probability of scattering or absorption for any given photon, as it passes through a thickness $\Delta x$ of material is the same as that for any other and we will also suppose that the photons don’t interact with each other, which is reasonable. Hence, the number of photons that are scattered is simply

$$n_{\text{scattered}} = n_0 p(\text{photon scattered}).$$  \hspace{1cm} (62)

Since the electric field strength in the wave is proportional to the number of photons, we can rewrite this equation as

$$\Delta E_0 = -(\alpha \Delta x) E_0$$  \hspace{1cm} (63)

and the probability has been replaced by $\alpha \Delta x$, with $\alpha$ a constant related to the properties of the medium. If we take the limit as $\Delta x \to 0$ and solve this differential equation, we obtain.

$$E_0(x) = E_0(0) e^{-\alpha x} = E_0(0)e^{-(\alpha_{\text{scat}} + \alpha_{\text{abs}})x},$$  \hspace{1cm} (64)

where we have divided $\alpha$ into a term related to scattering and term related to absorption.

**Question:** What are the units of $\alpha_{\text{scat}}$ and $\alpha_{\text{abs}}$?

Finally, we can insert this value of $E_0(x)$ into the solution to the 1-D wave equation and obtain

$$E(x, t) = E_0(0) e^{-\alpha x} = E_0(0)e^{-(\alpha_{\text{scat}} + \alpha_{\text{abs}})x}.$$
\begin{align}
E_0(0) e^{i(\omega t - (k - i(\alpha_{\text{scat}} + \alpha_{\text{abs}}))x)}
\end{align}

and so a commonly used way of describing the attenuation is by using a \textit{complex wavenumber} of form \( k - i\alpha \).

\section{Energy flow in waves}

\subsection{Comparison between wave motion and SHM}

When looking at the energy carried by a wave, it is important to be aware of some subtle differences from the situation of simple harmonic motion, which was analysed at Level 1. Let us think about waves in a string. Since each element \( \Delta x \) is a mass \( \rho \Delta x \) that is oscillating up and down, it is tempting to recall the results for a mass oscillating on a spring with vibrational amplitude \( a \). The SHM equation for this situation is

\begin{align}
m \frac{d^2y}{dt^2} = Ky,
\end{align}

where \( K \) is the spring constant, and the kinetic and potential energies are

\begin{align}
PE &= \frac{1}{2}Ky^2 = \frac{1}{2}Ka^2\cos^2\omega t \\
KE &= \frac{1}{2}m \left( \frac{dy}{dt} \right)^2 = \frac{1}{2}ma^2\omega^2\sin^2\omega t.
\end{align}

Since \( \omega = \sqrt{K/m} \), it is clear that the total energy is constant at all times. One might imagine that the mass of string oscillating up and down in a wave would be a bit like this, with the tension in the string playing a similar role to the spring constant. Let us look at the energy balance for an element of
Figure 20: Potential energy is stored in a string as it stretches

string. As with the SHM case, there is both kinetic and potential energy. You will not be surprised to find that

\[
d(KE) = \frac{1}{2} (\rho dx) \left( \frac{\partial y}{\partial t} \right)^2 = \frac{1}{2} (\rho dx) \omega^2 a^2 \sin^2(\omega t - kx) .
\] (68)

However, when we come to consider the potential energy, we get an unexpected result. The energy is stored by a force, the tension \( T \), moving through a distance, as it *stretches the string locally*. Look at Fig. 20. The string, originally flat, now has slope \( \partial y/\partial x \). It has stretched to accommodate this change in slope and the length has changed from \( dx \) to \( ds \), where

\[
ds = \sqrt{1 + \left( \frac{\partial y}{\partial x} \right)^2} \, dx .
\] (69)

As the string is stretched, work is performed on it and stored as elastic potential energy. The tension force \( T \) moves through a distance \( (ds - dx) \) and so the potential energy stored is

\[
d(PE) = T \, dx \left\{ \sqrt{1 + \left( \frac{\partial y}{\partial x} \right)^2} - 1 \right\} .
\] (70)

Using the binomial expansion on the square root term and ignoring any terms
higher than first order gives

\[ d(PE) = \frac{1}{2} (T \, dx) \left( \frac{\partial y}{\partial x} \right)^2 = \frac{1}{2} k^2 a^2 \sin^2(\omega t - kx) . \]  \hspace{1cm} (71)

The key thing to notice about this result is that the expression for the potential energy contains a \( \sin \) term and not a \( \cos \) term as in the SHM case. This means that when we add together the kinetic and potential energies, they do not come to a constant. Notice also that, whereas in the SHM case, the system has its maximum potential energy when the mass is displaced furthest from its equilibrium position, because that is where the spring is compressed or stretched the most, in the wave case, the maximum PE for the string element occurs when it is at its equilibrium position, because this is when the local gradient is highest and the string stretched the most.

Taking Eqs. (68) and (71) together and remembering that \( c = \sqrt{T/\rho} \), the overall energy density, i.e., energy per unit length is

\[ Tk^2 a^2 \sin^2(\omega t - kx) = \rho \omega^2 a^2 \sin^2(\omega t - kx) . \]  \hspace{1cm} (72)

Not all parts of the wave contain the same energy density at any given time and the positions of the energy maxima move forward with speed \( c \). If we take the average energy density, either spatially or temporally, it comes to \( \frac{1}{2} Tk^2 a^2 = \frac{1}{2} \rho \omega^2 a^2 \). Exercise: Check the derivation of Eq. (72) and do the appropriate integration to obtain the spatial and time averages.

### 8.2 Energy transport in different types of wave

We have just calculated the energy density in the wave. This leads us on to considering how much energy is actually carried. One simple way is to
say that if the average energy density in the wave is $\frac{1}{2}Tk^2a^2$ and the “length of wave” that passes through a given point per unit time is $c$ (definition of speed), then the total energy that passes through that point is simply $\frac{1}{2}Tk^2a^2c$.

For a different way of thinking about this, consider instead the current and voltage waves travelling waves travelling down a transmission line. We are familiar with the expression $W = VI$, which describes the power in an electrical circuit. In this case, $VI$ represents the instantaneous power at a particular point in the transmission line, i.e., the energy passing through a given point in the line per unit time.

Now let’s play our “analogy game” again, spotting that $V$ and $I$ are what I have called the conjugate variables for the transmission line system. Refer again to Table 2. The comparable variables for the vibrating string are $-T\partial y/\partial x$ and $u_y$. The product of these is

$$Tk a \sin(\omega t - kx) \cdot \omega a \sin(\omega t - kx) = (Tk)(ck) a^2 \sin^2(\omega t - kx),$$

since $\omega/k = c$. This is the instantaneous power passing through any given point and if we then, as before, take either the spatial or the temporal average, we obtain the same result as earlier, namely $\frac{1}{2}Tk^2c$. However, this method is perhaps more revealing than the previous one, as it has shown us how to calculate the instantaneous power at a given point, rather than just the average.

You should now be able to appreciate how powerful it is to be able to write down a general statement about waves and deduce from it specific formulae. Table 4 summarises various useful formulae for the power transport in the four different systems that we are studying. Notice that, simply by this analogy system, we have deduced the formulae for electromagnetic waves,
## Instantaneous power

<table>
<thead>
<tr>
<th>Wave Type</th>
<th>Instantaneous Power</th>
<th>Time av. power</th>
<th>Inst. energy density</th>
</tr>
</thead>
<tbody>
<tr>
<td>String</td>
<td>$-T \frac{\partial y}{\partial x} u_y$</td>
<td>$\frac{T^2}{2Z} \frac{(\partial y/\partial x)^2}{2Z}$</td>
<td>$\frac{1}{2} u_y^2 Z$</td>
</tr>
<tr>
<td>Sound</td>
<td>$pu_x$</td>
<td>$\frac{p_0^2}{2Z}$</td>
<td>$\frac{1}{2} u_x^2 Z$</td>
</tr>
<tr>
<td>Transmission Lines</td>
<td>$VI$</td>
<td>$\frac{V_0^2}{2Z}$</td>
<td>$\frac{1}{2} I_0^2 Z$</td>
</tr>
<tr>
<td>Electromagnetism</td>
<td>$E_y H_z$</td>
<td>$\frac{E_y \phi}{2Z}$</td>
<td>$\frac{1}{2} H_0^2 Z$</td>
</tr>
</tbody>
</table>

* assumes a sinusoidal waveform, whereas the instantaneous power expressions are general. For ultrasound waves, the instantaneous power transport is also called the intensity.

Table 4: Useful expressions for the power transported by various types of wave

something which you should derive more formally in the EM course.

**Exercise:** To make all these general formulae more relevant to everyday life, make rough order-of-magnitude estimates of the energy density, power transport and values of the conjugate variables for (a) a light bulb, (b) someone shouting, (c) a plucked guitar string, (d) the coax. cable in Experiment 1 of the Level 2 practical “Transmission Lines”.
Appendix

This appendix will not be examined, but contains material that will be useful for you in seeing how some of the equations from the main part of the course come about and in deepening your understanding of the concepts that have been introduced.

A 2-D waves on a stretched membrane

Let us consider a small section of a stretched membrane — a drum skin is a good example. Figure 21 shows the situation and also includes a projection of the membrane onto the \((x, y)\) plane. Notice that the projection is a rectangle with sides of length \(\Delta x\) and \(\Delta y\), but that if we look at the membrane itself, each of the corners has a different height, i.e., \(z = z(x, y)\), since the membrane is oscillating in the transverse direction. Shown in Figure 21 is the position at a single time, but this will change as the wave propagates. Hence, we actually want \(z = z(x, y, t)\).

Now the membrane is under a tension per unit length \(T\). If there is no tension on a drum skin, it doesn’t make a noise, i.e., no waves propagate. When no waves are present, i.e., \(z = 0\) everywhere, then a force diagram for the small area of membrane is as in Fig. 22(a) and (b) and all the forces balance. Notice in (a) that on all the sides of the square, the tension force is parallel to the local slope of the plane, in this case zero. In (c) and (d), the membrane is displaced and distorted by the passage of a wave. The tension forces are again parallel to the local slope at the membrane edge and this time, since the element is distorted, they no longer balance exactly. Let us resolve these forces in the \(x\), \(y\) and \(z\) directions respectively. Comparing the
Figure 21: Definition of variables for 2-D waves on a stretched membrane

Forces at AB and CD, we have:

\[ \text{Net } x-\text{force} = T \Delta y \cos \theta|_x - T \Delta y \cos \theta|_x \]
\[ \approx T \Delta y \left[ (1 + \theta_x^2 |x + \Delta x) - (1 + \theta_x^2 |x) \right] \quad (75) \]

\[ \text{Net } z-\text{force} = T \Delta y \sin \theta|_x - T \Delta y \sin \theta|_x \]
\[ \approx T \Delta y \left[ \theta_x |x + \Delta x - \theta_x |x \right] \quad (76) \]

Now if the oscillations are small, then \( \theta_x \) is small and \( \theta_x^2 \) will be entirely negligible. So, for the purposes of this argument, we can ignore the net force in the x-direction. As far as the z-force goes, we now note that, since the forces at AB and CD are parallel with the local slope of the plane, \( \tan \theta_x = \frac{\partial z}{\partial x} \). In our small displacement approximation, \( \theta_x \approx \tan \theta_x \approx \sin \theta_x \).

If the oscillations are not small, then neither of these assumptions will be
correct and the wave equation will take a more complicated form — see Sec. 3.3 for further details.

Exactly the same analysis applies to the tension forces along sides AC and BD. The overall force in the $z$-direction is thus given by

$$ F(x, y) = T \Delta y \left[ (\frac{\partial z}{\partial x})_{x+\Delta x} - (\frac{\partial z}{\partial x})_x \right] + T \Delta x \left[ (\frac{\partial z}{\partial y})_{y+\Delta y} - (\frac{\partial z}{\partial y})_y \right]. $$

(77)

You should recognise this expression. It is exactly the same as what we had when looking at waves on a string, except that now we have terms for both axes of propagation of the wave. The force produces an acceleration of the surface element ABCD, which is found once again using “F=ma”.

$$ T \Delta x \Delta y \left[ \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} \right] = (\rho \Delta x \Delta y) \frac{\partial^2 z}{\partial t^2}, $$

(78)

where $(\rho \Delta x \Delta y)$ is the mass of the surface element. This leads finally to the 2-D Wave Equation

$$ \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{\rho}{T} \frac{\partial^2 z}{\partial t^2} = \frac{1}{c^2} \frac{\partial^2 z}{\partial t^2}. $$

(79)
Notice that this equation is exactly the same as what we had earlier, except for the addition of a second term on the LHS, reflecting the existence of a second dimension.

**B  More on impedance**

**B.1 Derivation of the reflection formula**

The overall pressure variation in the medium can be described as the *sum* of two different pressure waves on the left of the interface (incident $I$, reflected $R$) and as a transmitted $T$ on the right of the interface:

\[
I = I_0 e^{i(\omega t - kx)}; \quad R = R_0 e^{i(\omega t - kx)}; \quad T = T_0 e^{i(\omega t - kx)} \quad (80)
\]

Although the derivation works for any position, let us consider an interface at $x = 0$, in which case, any $kx$ terms disappear. Two boundary conditions must be satisfied at the interface:

(i) Pressure is the same on either side of the interface, i.e., $P(0^-) = P(0^+)$. If this were not the case, then the interface as a whole would move. This means that

\[
I_0 e^{i\omega t} + R_0 e^{i\omega t} = T_0 e^{i\omega t}
\]

\[
\implies I_0 + R_0 = T_0 \quad (81)
\]

(ii) The particle velocity must be the same on either side of the interface, otherwise the two sides would pull away from each other, i.e., $u_x(0^-) = u_x(0^+)$. Writing $u_x$ in terms of $p$ and $Z$ and noting the minus sign for the reflected wave:
B. MORE ON IMPEDANCE

\[
\frac{I_0}{Z_1} - \frac{R_0}{Z_1} = \frac{T_0}{Z_2}
\]  

(82)

Solving these equations gives, as presented in the main text,

\[
R_0 = \frac{Z_2 - Z_1}{Z_2 + Z_1} I_0; \quad T_0 = \frac{2Z_2}{Z_2 + Z_1} I_0.
\]  

(83)

B.2 Reflection and Transmission: 3-D Treatment

The previous results are valid for normal incidence of the wave at the surface. For some other arbitrary angle of incidence, we need a 3-D version. (The following vectorial argument is couched in a slightly different form to what you will see in many textbooks, but comes to the same conclusions.)

We start off knowing nothing at all about what happens and see what we can deduce from just the wave equation and the two boundary conditions on pressure and velocity. Referring to Fig. 23 for the notation, let us rewrite Eq. [81] in vector form:

\[
I = I_0 e^{i(\omega t - k_i \cdot r)}
\]

Figure 23: Reflection and transmission at an interface studied in 3D
\[ R = R_0 e^{i(\omega_r t - k_r \cdot r)} \]
\[ T = T_0 e^{i(\omega_t t - k_t \cdot r)}. \]  

(84)

We have assumed nothing about the three waves other than that individually they are monochromatic and plane.

Our first boundary condition is that everywhere on the interface, pressure is continuous. Mathematically,

\[ I_0 e^{i(\omega_i t - k_i \cdot r)} + R_0 e^{i(\omega_r t - k_r \cdot r)} = T_0 e^{i(\omega_t t - k_t \cdot r)} \]

must be satisfied for all points \( \mathbf{r} = \alpha \mathbf{e}_\parallel + \beta \mathbf{e}_\perp \), where \( \mathbf{e}_\parallel \) and \( \mathbf{e}_\perp \) are the basis vectors which span the surface of the interface, which we assume is an infinite plane containing the origin. We also require that the equation is satisfied at all times \( t \). By trying a few examples, you should be able to convince yourself quickly that this can occur only if all the phases are the same. I.e.,

\[ \omega_i t - k_i \cdot \mathbf{r} = \omega_r t - k_r \cdot \mathbf{r} = \omega_t t - k_t \cdot \mathbf{r} \]  

(86)

If this to be valid at any given surface location, it must certainly be valid at \( \mathbf{r} = 0 \), in which case

\[ \omega_i t = \omega_r t = \omega_t t \]  

(87)

**Result 1**

The frequency of the reflected and transmitted waves is the same as that of the incident wave.
Now if the boundary condition is to be valid at any point $r = \alpha e_\parallel + \beta e_\perp$, then

$$k_i \cdot (\alpha e_\parallel + \beta e_\perp) = k_r \cdot (\alpha e_\parallel + \beta e_\perp) \quad \forall \alpha, \beta.$$  \hspace{1cm} (88)

However, $e_\perp$ was chosen to be perpendicular to the incident wave, so $k_i \cdot e_\perp = 0$. Hence,

$$k_i \cdot \alpha e_\parallel = k_r \cdot (\alpha e_\parallel + \beta e_\perp) = k_t \cdot (\alpha e_\parallel + \beta e_\perp).$$  \hspace{1cm} (89)

If this is to be true for any value of $\beta$, then $k_r \cdot e_\perp = k_t \cdot e_\perp = 0$

**Result 2**

The wavevectors of the incident wave, the reflected wave and the transmitted wave are all in the same plane.

Eq. [89] now becomes

$$k_i \cdot e_\parallel = k_r \cdot e_\parallel = k_t \cdot e_\parallel.$$  \hspace{1cm} (90)

Remember that the dot product depends both on the length of $k$ and the angle between $k$ and $e_\parallel$. So, as long as both change together, the equality can be satisfied. Now $\omega$ is constant throughout (see Eq. [87]) and we know that $\omega/|k| = c$ for each medium. Hence,

$$\frac{\omega}{c_1} k_i \cdot e_\parallel = \frac{\omega}{c_1} k_r \cdot e_\parallel = \frac{\omega}{c_2} k_t \cdot e_\parallel.$$  \hspace{1cm} (91)
The $\hat{k}$'s are unit vectors and so we have separated the angle from the length.

\[
\hat{k}_i \cdot e_\parallel = \hat{k}_r \cdot e_\parallel \tag{92}
\]

\[
\hat{k}_i \cdot e_\parallel = \frac{c_1}{c_2} \hat{k}_t \cdot e_\parallel \tag{93}
\]

Eq. [92] says that the cosine of the angle which the incident wave makes with $e_\parallel$ is the same as that for the reflected wave. (Note the directions of $k_i$ and $k_r$ in Fig. 12.) This is equivalent to:

**Result 3 (Law of Reflection)**

The angle of the incident wave to the normal is the same as the angle of the reflected wave to the normal.

Let $i$, $r$ and $t$ be angles with respect to the normal as in Fig. 12. Eq. [93] says that

\[
\cos(90^\circ - i) = \frac{c_1}{c_2} \cos(90^\circ - t). \tag{94}
\]

**Result 4 (Snell’s Law of Refraction)**

\[
\frac{\sin i}{\sin t} = \frac{c_1}{c_2} = \frac{Z_1}{Z_2} \quad (\text{since } Z = \rho_0 c) \tag{95}
\]

*From just the wave equation for pressure and the boundary condition that pressure is continuous, we have derived all the normal laws of reflection and refraction!* To finish the job and find the reflection and transmission amplitudes, we need to make use of one further boundary condition: particle velocity is continuous across the interface. From Section 5,
\[ \frac{I_0}{Z_1} \hat{k}_i + \frac{R_0}{Z_1} \hat{k}_r = \frac{T_0}{Z_2} \hat{k}_t \] (96)

Consider the velocity component perpendicular to the surface (i.e., parallel to \( \mathbf{n} \)).

\[ \frac{I_0}{Z_1} \hat{k}_i \cdot \mathbf{n} + \frac{R_0}{Z_1} \hat{k}_r \cdot \mathbf{n} = \frac{T_0}{Z_2} \hat{k}_t \cdot \mathbf{n} \] (97)

From Eq. [85], we know that \( I_0 + R_0 = T_0 \) and from Result 3 above, \( \hat{k}_i \cdot \mathbf{n} = -\hat{k}_r \cdot \mathbf{n} \). Hence,

\[ \frac{I_0 - R_0}{Z_1} \hat{k}_i \cdot \mathbf{n} = \frac{I_0 + R_0}{Z_2} \hat{k}_t \cdot \mathbf{n} \]

\[ \iff R_0 (Z_2 \hat{k}_i \cdot \mathbf{n} + Z_1 \hat{k}_t \cdot \mathbf{n}) = I_0 (Z_2 \hat{k}_i \cdot \mathbf{n} - Z_1 \hat{k}_t \cdot \mathbf{n}) \]

\[ \iff R_0 = \frac{Z_2 \hat{k}_i \cdot \mathbf{n} - Z_1 \hat{k}_t \cdot \mathbf{n}}{Z_2 \hat{k}_i \cdot \mathbf{n} + Z_1 \hat{k}_t \cdot \mathbf{n}} I_0 = \frac{Z_2 \cos i - Z_1 \cos t}{Z_1 \cos t + Z_2 \cos i} I_0. \] (98)

This is in agreement with Eq [46] for the case \( i = t = 0^\circ \).

Finally, it should be noted that this derivation applies strictly only for an infinite plane wave hitting an infinite plane wave surface. In real life, especially with ultrasonics, where the dimensions of reflecting objects start to approach the ultrasound wavelength, other effects, such as diffraction become important.
B.3 Proof of the Impedance-Matching Relation $Z_i = \sqrt{(Z_1 Z_2)}$

Consider the scenario shown in Fig. 24, in which a thin film of acoustic impedance $Z_i$ separates two materials with impedances $Z_1$ and $Z_2$. Let the wavelength and wavenumber of the radiation in the film be $k$ and $\lambda$ respectively, such that the phase shift caused by propagation a distance $d$ in the film is $kd$.

Let us apply our normal boundary conditions on pressure and velocity at the two points $x = 0$ and $x = d$. As in Eq. [81], we may divide both sides by $e^{i\omega t}$, leaving

\[ I_0 + R_0 = T_0 + R_1 \quad \text{(99)} \]

\[ Z_1(I_0 - R_0) = Z_i(T_0 - R_1) \quad \text{(100)} \]

from the conditions at $x = 0$ and

Figure 24: Illustration of impedance matching in a diagnostic ultrasound probe
$$T_0 e^{-ikd} + R_0 e^{+ikd} = T_1 e^{-ik'd}$$

(101)

$$Z_i (T_0 e^{-ikd} - R_0 e^{+ikd}) = Z_2 T_1 e^{-ik'd},$$

(102)

where $k'$ is the wavenumber in medium 2. Its exact value is unimportant and from now on, we shall replace $T_1 \exp(-ik'd)$ by the complex constant $A$.

We now use Eqs.[99] and [100] to eliminate $T_0$ and $R_1$:

$$[99] + \frac{1}{Z_i}[100] \rightarrow T_0 = \frac{1}{2} I_0 \left(1 + \frac{Z_1}{Z_i}\right) + \frac{1}{2} R_0 \left(1 - \frac{Z_1}{Z_i}\right)$$

(103)

$$[99] - \frac{1}{Z_i}[100] \rightarrow R_1 = \frac{1}{2} I_0 \left(1 - \frac{Z_1}{Z_i}\right) + \frac{1}{2} R_0 \left(1 + \frac{Z_1}{Z_i}\right).$$

(104)

Substitute these values into Eqs. [101] and [102] and rearrange. You will see appearing a whole set of pairs of the form $\frac{1}{2} I_0 (e^{+ikd} \pm e^{-ikd})$, which can be converted into $I_0 \cos kd$ and $I_0 i \sin kd$. After rearrangement,

$$I_0 [\cos kd - \frac{Z_1}{Z_i} i \sin kd] + R_0 [\cos kd + \frac{Z_1}{Z_i} i \sin kd] = A$$

(105)

$$I_0 [-i \sin kd + \frac{Z_1}{Z_i} \cos kd] + R_0 [-i \sin kd + \frac{Z_1}{Z_i} \cos kd] = Z_2 A.$$  

(106)

The trick now is to solve these simultaneous equations by multiplying Eq. [105] by $i \sin kd$ and [106] by $\cos kd$. All the sin cos terms cancel, whilst things in $\sin^2 + \cos^2$ add to 1. Hence,

$$\frac{Z_1}{Z_i} I_0 + \frac{Z_1}{Z_i} R_0 = A \cdot i \sin kd + Z_2 A \cos kd.$$  

(107)
Tidying this up a bit gives

\[ Z_1(I_0 - R_0) = (Z_1 \cdot i \sin kd + Z_2 \cos kd) A \] \hspace{1cm} (108)

Similarly, \( [106] \times \cos kd + [105] \times i \sin kd \) leads to

\[ I_0 + R_0 = (\cos kd + \frac{Z_2}{Z_i} i \sin kd) A \] \hspace{1cm} (109)

Finally, for perfect impedance matching, \( R_0 = 0 \) — i.e., no reflected wave from the first boundary. This happens for \( d = \lambda/4 \iff kd = \pi/2 \), which leads to

\[ \frac{Z_1}{Z_i} I_0 = iA \] \hspace{1cm} (110)

from Eq. [108] and

\[ I_0 = \frac{Z_2}{Z_i} iA \] \hspace{1cm} (111)

from Eq. [109]. Together, these lead straight to the condition

\[ \frac{Z_1}{Z_i} = \frac{Z_i}{Z_2} \iff Z_i = \sqrt{Z_1Z_2}. \] \hspace{1cm} (112)

### B.4 The specific impedance associated with stationary waves

A stationary wave pattern is formed when two plane waves travelling in opposite directions interfere.

\[ V = V_+ + V_- = V_0 \left[ e^{i(\omega t-kx)} + e^{i(\omega t+lx)} \right]. \] \hspace{1cm} (113)
The current corresponding to $V_+$ is $I_+ = V_+/Z_0 = V_+/L_0c$. If you follow the same procedure as was used to derive this, but for the backward-travelling wave, you should obtain $I_- = -V_-/L_0c$, i.e., $Z = -L_0c$ for this wave. The specific impedance for the total wave is thus

$$Z = \frac{V}{I} = \frac{V_+ + V_-}{I_+ + I_-} = \frac{V_0}{L_0c} \left[ e^{i(\omega t - kx)} + e^{i(\omega tpkx)} \right]$$  \hspace{1cm} (114)$$

We now expand the complex exponentials into sines and cosines to give

$$Z = L_0c \frac{e^{i\omega t} \left[ e^{-ikx} + e^{+ikx} \right]}{e^{i\omega t} \left[ e^{-ikx} - e^{+ikx} \right]} = L_0c \frac{2 \cos(-kx)}{2i \sin(-kx)} = iL_0c \cot(kx) \hspace{1cm} (115)$$

This odd result needs some thinking about!

(a) The impedance is complex. This indicates that no net energy is transported by the wave in question. This is reasonable given that $I$ is out of phase with $V$. Here equal quantities of energy are being transported in opposite directions. We know that the power is $W = VI$ and so

$$W = \text{Re} \left[ e^{i\omega t} \cdot 2V_0 \cos(-kx) \right] \cdot \text{Re} \left[ e^{i\omega t} \frac{2iV_0}{L_0c} \sin(-kx) \right]$$

$$= \cos \omega t \cdot 2V_0 \cos kx \cdot \sin \omega t \cdot \frac{2V_0}{L_0c} \sin kx$$

$$= \frac{V_0^2}{L_0c} \sin 2\omega t \cdot \sin 2kx$$ \hspace{1cm} (116)$$

The time averaged power $\langle W \rangle = 0$.

(b) The impedance varies spatially. The nodes and antinodes of the voltage and current stationary wave patterns are exactly $90^\circ$ out of phase with each other. Wherever $V$ has a node, $I$ has an antinode. It is clear that the ratio of $I$ to $V$ will depend on where on the line you are and can be anything between 0 and $\infty$. 
Notice, finally, that the spatial average of power transport in the wave pattern at a fixed time is

$$W_{av} \propto \int \sin 2kx \, dx = 0 \quad \text{over a whole no. of cycles.} \quad (117)$$

Power in the stationary wave pattern is moving both backwards and forwards along the line at different sites and at different times, but the net movement, both spatially and temporally is zero. The same will be true for any stationary wave pattern, e.g., sound, optics, etc.

**B.5 3-D example — spherical sound waves**

Consider again our spherical sound wave, given by $p = (A/r^2)e^{i(\omega t - kr)}$. What is $Z$ in this case? To answer this, we need to take a closer look at what “longitudinal waves in 3D” are. Our particle displacement $\eta$ is actually a 3-D vector and the appropriate wave equation is

$$\nabla^2 \eta = \frac{1}{c^2} \frac{\partial^2 \eta}{\partial t^2} \quad (118)$$

This is an incredibly compact way of writing three separate equations of form

$$\frac{\partial^2 \eta_x}{\partial x^2} + \frac{\partial^2 \eta_y}{\partial y^2} + \frac{\partial^2 \eta_z}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 \eta_x}{\partial t^2}, \quad (119)$$

where there is a similar equation for $\eta_y$ and $\eta_z$. Following this through to its logical conclusion, Table 2 should have three entries for sound waves:

$$\frac{\partial p}{\partial x} = -\rho_0 \frac{\partial u_x}{\partial t}; \quad \frac{\partial p}{\partial y} = -\rho_0 \frac{\partial u_y}{\partial t}; \quad \frac{\partial p}{\partial z} = -\rho_0 \frac{\partial u_z}{\partial t}. \quad (120)$$

Notice that $p$ is a scalar and does not have components. Always in the market for something to reduce the number of equations to be learnt, let’s
put all this in vector form. Suppose we write \( u = u_x i + u_y j + u_z k \) for the general particle velocity. Then

\[
\frac{\partial u}{\partial t} = \left( \frac{\partial u_x}{\partial t} \right) i + \left( \frac{\partial u_y}{\partial t} \right) j + \left( \frac{\partial u_z}{\partial t} \right) k
\]

\[= -\frac{1}{\rho_0} \left[ \left( \frac{\partial p}{\partial x} \right) i + \left( \frac{\partial p}{\partial y} \right) j + \left( \frac{\partial p}{\partial z} \right) k \right]. \quad (121)\]

The quantity in brackets has the special name \( \nabla p \) and so our three equations are reduced to

\[
\nabla p = -\rho_0 \frac{\partial u}{\partial t} \quad (122)
\]

Just as with the \( \nabla^2 \) operator described earlier, \( \nabla \) represents an “operation in space” and has an “existence” that is independent of the co-ordinate system we choose. See Boas for more details. For a spherical wave, the pressure variations are entirely radial and, in this simplified case, the spherical polars representation of \( \nabla \) reduces to simply

\[
\nabla = \hat{r} \frac{\partial}{\partial r} \quad (123)
\]

Substituting this into Eq (122) shows that the particle acceleration and hence velocity and displacement are in the radial direction. This conforms with a definition of a longitudinal wave as one where the particles vibrate parallel to the (local) direction of propagation. Notice that this is different everywhere. Reflecting this, there is no \( k \) vector associated with the wave, as with plane waves. To conclude, we note that the definition of specific impedance for a 3-D sound wave is \( Z(r) = p(r, t)/|u(r, t)| \) and proceed as in Problem 4.
C  MOMENTUM CARRIED BY A WAVE: RADIATION PRESSURE 67

C Momentum carried by a wave: radiation pressure

There is a very general relationship that you will find in many branches of Physics, which says that a flow of energy via a wave motion brings with it also a flux of momentum. If we use \( p \) as is conventional to represent the momentum — be careful to distinguish this from the \( p \) for acoustic pressure —, then the general relationship is

\[
p = \frac{E}{c}
\]  

(124)

For an ultrasound beam of area \( A \), the total energy carried in one second is \( A\langle I \rangle \), where \( \langle I \rangle = \langle pu_x \rangle \) is the time-average intensity. The momentum carried per second is thus \( A\langle I \rangle/c \). Now Newton’s 2nd Law tells us that a change in momentum leads to a force, so that this means that the ultrasound beam leads to a force on the area \( A \) of

\[
F = \frac{A\langle I \rangle}{c}
\]  

(125)

and since pressure = force / unit area, the ultrasound beam is exerting a pressure \( P_{rad} = \langle I \rangle/c \). This is commonly known as radiation pressure. Note the difference between this and the ambient atmospheric pressure \( P \). If, instead of being absorbed, the ultrasound is reflected, then the momentum changes from \( p \) to \(-p\) (i.e., the total change is \(2p\)) and so the pressure is doubled. This pressure is extremely small, but may affect delicate structures in the body.

Feynman gives an interesting discussion of radiation pressure in two sections of his Lectures on Physics (Sec. I-34-9 and Sec. II-27-6). The application of these formulae to light provides a very neat “tie-up” between the wave the-
D DISPERSION

ory of light and quantum mechanics. You should learn in other courses that the complete version of Einstein’s famous “\( E = mc^2 \)’’ equation is

\[
E^2 = m_0^2 c^4 + p^2 c^2 .
\]  

(126)

For a photon with zero rest mass, this reduces simply to \( E = pc \) and this is the same as Eq. (125). See Pain Ch. 13 for further details of how the wave and particle theories are related.

Exercise: Estimate the radiation pressure from (a) a 2 W cm\(^{-2}\) ultrasound beam and (b) a 100 W light bulb.

D Dispersion

You should by now be very used to the idea of phase velocity. This is what we commonly think of as “the speed of a wave” and it is the constant \( c \) that appears in the Wave Equation. For many problems, however, this “constant” is not constant at all, but varies with frequency. We know that \( c \) figures in the equation \( c = f \lambda \) or, equivalently, \( c = \omega/k \) and a variation in \( c \) must mean that the relationship between \( \omega \) and \( k \) (or \( f \) and \( \lambda \)) is not the linear one that you might at first expect. Fig. 25 shows an example of such a dispersion relation.

Dispersion is an important phenomenon, because it causes wave groups to change their shape as they propagate. To understand this, let us start by considering a superposition of two sinusoidal waves, which we will represent as

\[
\psi_1(x, t) = a \cos(\omega_1 t - k_1 x) \quad \text{and} \quad \psi_2(x, t) = a \cos(\omega_2 t - k_2 x) \quad (127)
\]
Figure 25: Example of a dispersion relation. This is the relationship appropriate to water waves and includes a surface tension term.

The combined wave motion has the form

\[ \Psi = \psi_1 + \psi_2 = 2a \cos \left[ \frac{(\omega_1 - \omega_2)t}{2} - \frac{(k_1 - k_2)x}{2} \right] \cos \left[ \frac{(\omega_1 + \omega_2)t}{2} - \frac{(k_1 + k_2)x}{2} \right] \]

(128)

We can regard this as a wave motion with the average frequency of \( \psi_1 \) and \( \psi_2 \), i.e., \( \omega_1 + \omega_2 \), but amplitude-modulated by a much lower frequency wave with \( \omega_{\text{modulation}} = \omega_1 - \omega_2 \). The two types of oscillation move with speeds \( c_{\text{wave}} \) and \( c_{\text{modulation}} \), where

\[ c_{\text{wave}} = \frac{(\omega_1 + \omega_2)}{(k_1 + k_2)} \quad \text{and} \quad c_{\text{modulation}} = \frac{(\omega_1 - \omega_2)}{(k_1 - k_2)}. \]

(129)

Now if \( \omega \propto k \), it is clear that both of these speeds will reduce to a single constant \( c \), which is the slope of the \( \omega \) v. \( k \) relation. However, if the relationship is more complicated, as shown in Fig. 25, then the two are not equal. Let us consider a particular case, where the two waves \( \psi_1 \) and \( \psi_2 \) are very close in frequency. The speed of the modulations (or “groups” as they are often called) is \( \Delta \omega / \Delta k \). In the limit where \( \Delta \omega \to 0 \), we obtain the so-called group velocity. The speed of the “main” wave is called the phase velocity,
what we have previously called $c$. Thus,

$$v_{\text{phase}} = c = \frac{\omega}{k} \quad \text{and} \quad v_{\text{group}} = \frac{\partial \omega}{\partial k} \quad (130)$$

For an application of this to plasmas, see Pain.

An important consequence of dispersion (and the reason that it is called this!) is that if we have a complicated wave shape that can be Fourier analysed into many frequency components, then each of those components will potentially travel at a different speed. The different components can get out of sync with each other and a sharp wave group can “disperse” itself. Problem 6 involves simulating this phenomenon and watching the shape of the wave change as it propagates.