Simulating the quantum Zeno and anti-Zeno effects during position measurements

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We present a simple model for the time evolution of a quantum wave packet tunnelling through a potential barrier. We then simulate the process of repeated position measurements and clarify the conditions under which one observes either a quantum Zeno effect (the inhibition of a quantum system's dynamic evolution through repeated idealised measurements) or an anti-Zeno effect (an enhancement or speeding up of the quantum system's dynamic evolution due to its interaction with the measuring device).

One of the outstanding problems in the foundations of quantum mechanics is that of quantum measurement. In the framework of standard quantum measurement theory this is defined according to von Neumann [1]: a measurement projects the wavefunction of the system onto one of the eigenstates of the operator corresponding to that observable. But since any interaction between the quantum system and the measuring apparatus is always very complex the von Neumann postulate is rather hard to quantify. Nevertheless, a number of attempts have been made over the years and we will here revisit one consequence of the measurement process: the quantum Zeno effect (QZE). This describes the inhibition of the system's dynamic evolution when it is subjected to ideal measurements. More specifically, the Zeno effect states that a continuous meaurement will repeatedly collapse the state of the system onto its initial one thus preventing it from evolving at all. Indeed, as stated by Dirac [2]: "When we [make a measurement], the disturbance involved … causes a jump in the state of the dynamical system. From physical continuity, if we make a second measurement of the same observable immediately after the first, the result of the second measurement must be the same as that of the first".

A number of authors have investigated the quantum Zeno effect, most notably Misra and Sudarshan [3] who considered the spontaneous emission process of an unstable particle and claimed that a continuously observed state will never decay. More recently work has attempted to model the measuring device explicitly [4, 5] in which the measurement process entangles the state of the system with that of the measuring device causing a decay of the off-diagonal elements of the system's density matrix. Of interest in the present paper is the numerical simulation of the Zeno effect such as has been described by several authors recently [6, 7] who consider a particle that starts off on one side of a double well potential and investigate the effects of measurement on the time it takes for the particle to tunnel between the wells. However, the authors come to somewhat conflicting conclusions as to whether continuous measurement slows down or speeds up the tunnelling process. The latter is known as the anti-Zeno effect (AZE). For instance, the standard argument is that repeated measurement continually collapses the state of the particle with overwhelming likelihood back to its initial state on one side of the barrier. On the other hand, it is acknowledged that the act of measurement unavoidably disturbs the system, giving it repeated kicks which enhance the probability of barrier penetration. However, these models are relatively complex and make it difficult to see what is happening in a transparent way. Wallace [8] chose to consider a much simpler example: the evolution of a one-dimensional wavepacket describing the motion of a free particle. He considers the time evolution of the density operator in momentum space then Fourier transforms it back to configuration space before simulating the measurement process by setting all off-block-diagonal elements to zero. He concluded that the effects of repeated measurement can have non-trivial dynamical effects both on the rate of the spreading of the wavepacket and on rate of its centre-of-mass motion, sometimes speeding it up and sometimes slowing it down. But since he only dealt with free particles he could say nothing about the effects of the measurement on quantum tunnelling.

In this work, we extend the idea of Wallace to the more interesting case of the tunnelling of a wavepacket through a square potential barrier. In this way we are able to investigate the more realistic example of particle decay, which is closer in spirit to the models of refs. [3, 6, 7]. We are able to draw clear conclusions which resolve the dispute as to whether tunnelling is hindered or enhanced through repeated measurement. While our model is only solvable numerically, it is beautifully simple and transparent. Rather than attempting to solving the time-dependent Schrödinger equation we instead consider the time evolution of the density matrix directly via the von Neumann equation.

We begin by defining the density operator at time t as

$$\hat{\rho}(t) = |\psi(t)\rangle\langle\psi(t)|, \qquad (1)$$

where its matrix elements in 1-D coordinate space representation are

$$\rho(x,x',t) = \langle x|\hat{\rho}(t)|x'\rangle = \psi(x,t)\psi^*(x,t) \qquad (2)$$

and its time evolution is described by an equation of motion [9]

$$i\hbar\frac{\partial\hat{\rho}}{\partial t} = [H,\hat{\rho}] \qquad (3)$$

where H is the Hamiltonian operator for the system. The above equation is analogous to the Liouville equation for the variation of the classical probability density in phase space. We define the wavefunction of the system as a normalised 1-D Gaussian wavepacket at t = 0

$$\psi(x,0) = C\exp((x-x_0)^2/\omega^2 + ikx) \qquad (4)$$

with normalization constant $C=(\omega^2\pi/2)^{-1/4}$. We then follow the method of Wallace [8] and coarse grain the position in to a discrete lattice of position states. We define the basis of position eigenstates as $\{|X_n\rangle, n=1\ldots N\} \equiv |X_1\rangle, |X_2\rangle, \ldots, |X_n\rangle$ so that all quantities are represented on a grid at uniformly-spaced points X_i. We will consider only local potentials, i.e. $V_i \equiv V(X_i)$ does not depend on physical quantities at other grid points. The density matrix elements, $\rho_{ij}(t)$, therefore satisfy the equation

$$i\hbar\frac{\partial\rho_{ij}}{\partial t} = \langle X_i|[H,\hat{\rho}]|X_j\rangle. \qquad (5)$$

Inserting a complete set of states $\Sigma_k|X_k\rangle\langle X_k|$ into each term in the commutator in the above equation, we have

$$i\hbar\frac{\partial\rho_{ij}}{\partial t} = \sum_k (H_{ik}\rho_{kj} - \rho_{ik}H_{kj}), \qquad (6)$$

where the Hamiltonian matrix elements are

$$H_{ij} = \langle X_i|H|X_j\rangle = \left(\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial X_i^2} + V_i\right)\delta_{ij}. \qquad (7)$$

For ease of computation we then approximate the second-order derivative in the kinetic energy operator using the three-point formula

$$f''(X_i) \approx f(X_{i-1}) - 2f(X_i) + f(X_{i+1}) \qquad (8)$$

where the grid spacing has been defined to have unit length such that $\Delta X = X_i - X_{i-1} = 1$. Equation (7) then simplifies through the use of (8) and the fact that the potential is diagonal, to

$$i\frac{\partial\rho_{ij}}{\partial t} = \rho_{i-1,j} + \rho_{i+1,j} - \rho_{i,j-1} - \rho_{i,j+1} + (V_i - V_j)\rho_{ij}, \qquad (9)$$

where we have set, for convenience, $\hbar=1$, $m=1/2$. The above first order differential equation in time is solved using Runge-Kutta algorithm for the full $N \times N$ coupled equations (one for each element in the density matrix).

Although the choice of the form of the potential is arbitrary, since our method can cope with any shape, we consider here just a simple square barrier placed at the centre of the box and having an appropriately chosen width and height. The wavefunction at t = 0 is that of a particle starting off on the left of the barrier, which propagates in the positive direction. According to standard quantum mechanics, at any given time t the values of the diagonal elements ρ_{ii} correspond to the probabilities of finding the particle at grid points i. Off-diagonal elements correspond to non-physical probabilities and an ideal measurement sets all these elements to zero. As such, the density matrix is said to have decohered from a pure state to a mixed one, with its diagonal elements corresponding to classical probabilities for the various possible results of measurement. This process of decoherence removes the non-physical “Schrödinger cat” states. While decoherence does not explain how just one of the remaining physically observable outcomes is then ‘picked out’ from the statistical ensemble, we find that we need not concern ourselves with this aspect of the measurement problem since we present our results by plotting, at appropriate intervals of time, the full position probability density (the diagonal elements of the density matrix) for the particle. These plots do not of course correspond to measurements themselves but simply show the probability distribution in the particle’s position prior to a position measurement being made at that time.

We consider a measurement to have taken place when the wavefunction of the particle becomes coupled with that of the measuring apparatus. We follow the standard approach first described by von Neumann [1] and later developed by Arthurs and Kelly [10] and Fearn and Lamb [6]. The wave function of the measuring apparatus can be described by a simple wavepacket (although we do not need to define it explicitely) with a width that defines the position resolution of the measurement, which we refer to as the pointer width. Since we have discretised the position eigenstates onto a lattice of grid points then the pointer width cannot be smaller than the distance between two adjacent lattice points X_i and X_{i+1}. However, the position resolution of the measurement will in general be larger than this. Indeed, for the case of a particle tunnelling through a potential barrier, we are interested simply in whether the tunnelling has taken place or not (so as to make an analogy with radioactive decay, say, where our detector can only tell us if the nucleus has decayed or not). In this case, the pointer width is N/2 grid points.

In general, for a pointer grid size P, the position of the particle can only be resolved into N/P possible locations. We show results for two different pointer sizes corresponding to N/P = 2 (where we can only tell if the particle has tunnelled through the barrier or not) and N/P = 6. In each case, the density matrix elements can then be grouped into blocks of size P × P to give a new N/P × N/P block matrix. A measurement at time t is then simulated by the act of setting all off-diagonal blocks to zero since our pointer does not allow us to resolve the diagonal any more accurately.

The various parameters we choose for the initial wave function (width, starting position and momentum) are chosen to give a propagation speed and spread that is appropriate within the chosen grid of N = 300 positions such that it reaches the other side of the box within a few hundred time steps. The potential barrier is then chosen in order that a majority of the wavepacket is transmitted so that we may be able to see a clear zeno effect. The left hand column in Fig. 1, labelled ‘(a)’ shows this time evolution in the absence of measurements (we show only every 100th time frame, as labeled). We now allow the wave packet to propagate, but begin a series of successive measurements of its position, according to the prescription in the last section, beginning just before it reaches the barrier (at time t = 150).

We begin, as mentioned, with the most realistic pointer size of half that of the entire box (P = 150) which divides up the possible positions into just two: the particle can be either to the left or the right of the barrier. With such a low position resolution we restrict the momentum transfered to the wave function caused by the measurement. This is equivalent to a low energy, long wavelength, photon used to probe the location of an electron, say, and being unable to resolve its position to very high accuracy, but on the other hand not imparting a large ‘kick’ to the electron and disturbing its evolution. We will return to this issue later.

Column (b) of Fig. 1 shows the effect of measurements repeated at frequent intervals. A P=150 pointer measurement is made every one unit of time. By t=400 hardly any of the wavepacket has been allowed to tunnel through the barrier. We see here a very clear Zeno effect. Several points should be mentioned though. Firstly, since we are setting to zero off-diagonal pointer-sized blocks in the density matrix, this has the effect of removing flux and total position probability is no longer conserved. Since we must have a total unit probability at all times (the particle must be in the box), we always present are normalized probability in the figures. Secondly, if you look closely it

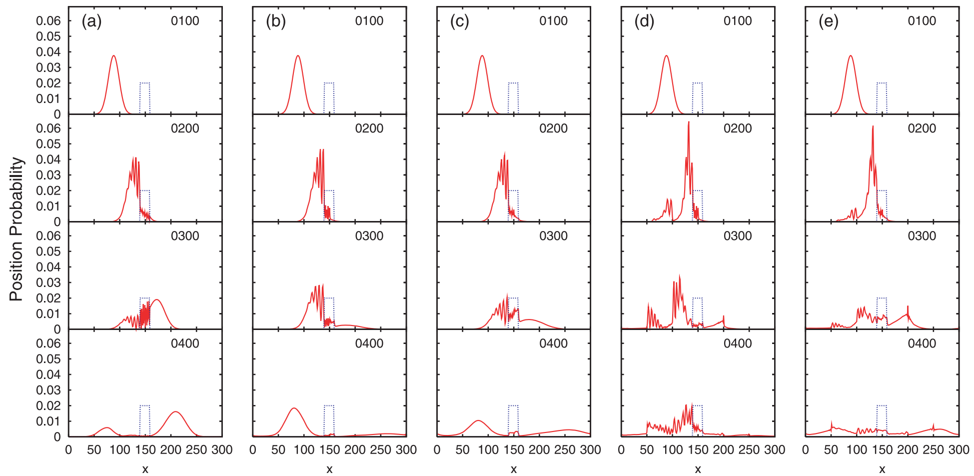

Figure 1: Time evolution of the wavepacket in the presence of a square barrier potential with and without repeated position measurements. The numbers in each frame indicates the time during the simulation. Column (a) is in the absence of any measurements. Column (b) show measurements carried out with a pointer size of P=150 dividing the grid into two halves. Measurements begin at t=150 and then are made at successive unit time steps until the end. A clear QZE is seen. Column (c) is the same as (b) but with measurements carried out every 4th time step. The QZE is weaker as tunnelling can now take place in between measurements. Column (d) is as for (b) but with a reduced pointer size of P=50 splitting the box into six pointer position states. Column (e) is as for (c) but with P=50. The increased momentum transfer associated with a smaller pointer width in (d) and (e) spreads the wavepacket out more.

is clear that some of the wave function has managed to tunnel through the barrier, implying that the Zeno effect is not perfect. However, our procedure of measuring once per unit time is not the same as continuous measurement. The Runge-Kutta algorithm for solving the first order differential equations (9) works by evaluating the function at each consecutive step based on its value and its gradient at the mid-point between two successive steps. Thus, we are also evaluating the density matrix elements at half-time-step intervals and so our successive measurements are not truly continuous. These half steps allow off-diagonal elements to have non-zero values in between measurements and this then 'contaminates' the diagonal elements corresponding to positions on the right of the barrier. Ideally, there should be no tunnelling at all if the measurements are continuous.

In column (c) of Fig. 1, we investigate this further by deliberately only taking measurements every four time steps. In between measurements it is clear that the wave function can indeed tunnel through but not to the extent that it could if completely left alone. Therefore with this large pointer size, we show a clear QZE that is enhanced as the measurements become more frequent.

We then investigate the effect of reducing the pointer size to allow for higher resolution in the position. We choose a value of P = 50 to divide up the box into six segments shown in columns (d,e) of Figs. 1. To begin with, we carry out a measurement at each time step. The results are shown in column (d) and should be compared with the results with the larger pointer size in (b). Again we see a clear QZE. The increased oscillations in this case are expected. What is happening essentially is that each measurement is forcing a separation of the full wave function into six independent components each localized in its own pointer segment. This has the same effect as placing infinitely high potential barriers on the boundaries between the segments. This causes the parts of the wavepacket, previously resolved into components of the free space, plane wave eigenstates, to be recast in eigenstates of the infinite square well potential, with higher energy eigenvalues and of an oscillatory nature. This effect also accounts for the resulting smearing out of the wavepacket, since during the measurement process, or equivalently the application of the infinite barriers between the pointer segments, some part of the wavepacket is given a higher energy, causing a greater dispersion as the system evolves in time.

To highlight the AZE described by Fearn and Lamb [6], we alter the barrier height until most of the wavepacket is reflected. Here we can show more clearly how a reduced pointer size can hasten decay (AZE) by transferring momentum to the particle, despite momentarily localizing it. Of course, these two effects are related by the uncertainty principle. The combination of less frequent measurement and smaller pointer width causes an increase in the amount of the wavepacket which passes through the barrier by the end of the simulation compared with no measurement, and is hence an example of the AZE. Fig. 2 shows the outcome at t = 400 of simulations for different frequencies of measurement, as indicated. In the absence of measurement (a), the fraction of probability to the right of the barrier is 0.13. With observations every one (b), four (c), and sixteen (d) timesteps, the probability of finding the particle to the right of the barrier is 0.04, 0.14 and 0.23 respectively, showing cases in which the Zeno effect occurs (measurement every unit of time), the anti-Zeno effect occurs (measurement every 16 units of time), or where the tension between the Zeno and anti-Zeno effect cancels (meausrement every four units of time).

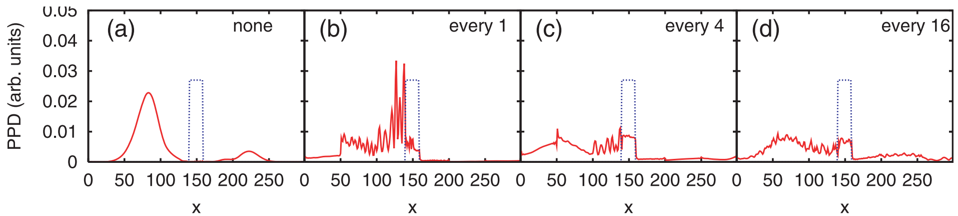

Figure 2: The position probability density (diagonal elements of the density matrix) at t=400 for no measurement, one measurement per unit time, one measurement every four units of time and one measurement every 16 units of time.

We have presented a simple model for the time evolution of a 1-D wave packet through a potential barrier by solving the equation of motion of the density matrix rather than the time-dependent Schrödinger equation. We simulate the process of position measurement at any given time for a certain choice of position resolution by setting to zero the off-diagonal terms in the density matrix.

We have clarified in a very transparent way the conditions under which one observes either a QZE or an AZE We have shown that which effect takes place depends on the type of measurements that are made. Rapid measurements made by a device with low position resolution will cause a clear Zeno effect that decreases as the frequency of the measurements decreases. But if the device has a higher position resolution then it unavoidably transfers momentum to the system These kicks cause it to evolve more rapidly provided the measurements are infrequent enough to reduce the prohibiting effect.

References

1. J. von Neumann, Mathematical Foundations of Quantum Mechanics Princeton University Press, Princeton, NJ, (1955)
2. P.A.M. Dirac, The Principles of Quantum Mechanics, Oxford University Press, Oxford (1930)
3. B. Misra and E.C.G. Sudarshan, J. Math. Phys. 18, 756 (1977)
4. A. Peres, Quantum Theory: Concepts and Methods Kluwer Academics Publishers, Dordretch (1993)
5. L. Schulman, Found. Phys. 27, 1623 (1997)
6. H. Fearn and W. E. Lamb Jr., Phys. Rev. A 46, 1199 (1992)
7. T.P. Altenmüller and A. Schenzle, Phys. Rev. A 49, 2016 (1994)
8. D. Wallace, Phys. Rev. A 63, 022109 (2001)
9. J.M. Ziman, Elements of Advanced Quantum Theory Cambridge University Press, Cambridge (1969).
10. E. Arthurs and J. Kelly, Bell. Syst. Tech. J. 44, 725 (1965).