Why 4-D?

It has already been claimed that SR requires time as the 4th dimension but we have not yet discussed why this is necessary. After all, any event requires 4 numbers to define it (x,y,z,t). Doesn't this already imply that time is an extra dimension or necessary axis. Also, the claim that the L.T. equations mix up space and time coordinates is not so convincing since the G.T.s do too ($x' = x - vt$). However, the real difference, that puts time on a different footing to what Galileo and Newton would have claimed is the transformation equation for time itself. It is no longer the same in different frames. What this means is that space and time cannot be split any longer into separate quantities but are mixed up in a very real and profound way.

Consider a length in 3-D space defined in terms of the coordinates of its end points (x_1,y_1,z_1) and (x_2,y_2,z_2). This is the space interval we met earlier:

$$l^2 = \Delta x^2 + \Delta y^2 + \Delta z^2$$

Transforming to a different reference frame *S'*, moving at velocity *v* in the x-direction, the Galilean Transformation equations would tell us that $l'^2 = l^2$ (since there is length contraction here) and we have invariance of length in 3D space. But when we use L.T. equations we know that $\Delta x' = \gamma \Delta x$ and so $l'^2 \neq l^2$.

Consider two events defined in frame *S* by the coordinates (x_1,y_1,z_1,t_1) and (x_2,y_2,z_2,t_2). Now define a quantity

$$\begin{aligned} s^2 &= (t_2 - t_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2 \\ &= \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2 \end{aligned}$$

To ensure the relation is dimensionally correct, we multiply the first term by c^2 (the reason we use the speed of light here rather than some other quantity with units of speed will become apparent later):

$$s^2 = c^2 \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$$

Now we work out what this quantity is (some kind of space and time distance between the two events) when measured in the *S'* frame:

$$\begin{aligned} s'^2 &= c^2 \Delta t'^2 - \Delta x'^2 - \Delta y'^2 - \Delta z'^2 \\ &= c^2 \gamma^2 \left(\Delta t - v\Delta x / c^2\right)^2 - \gamma^2 \left(\Delta x - v\Delta t\right)^2 - \Delta y^2 - \Delta z^2 \\ &= \gamma^2 \left[c^2 \Delta t^2 - 2v\Delta x \Delta t + \frac{v^2}{c^2} \Delta x^2 - \Delta x^2 + 2v\Delta x \Delta t + v^2 \Delta t^2 \right] - \Delta y^2 - \Delta z^2 \\ &= \frac{1}{1 - \frac{v^2}{c^2}} \left[\left(c^2 - v^2\right) \Delta t^2 + \left(1 - \frac{v^2}{c^2}\right) \Delta x^2 \right] - \Delta y^2 - \Delta z^2 \\ &= c^2 \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2 \\ &= s^2 \end{aligned}$$

So, s^2 is invarient under L.T.
s is known as the **spacetime interval.**

So, even though distances between events and times between events change when we go from one frame to unother under L.T., there exists a quantity (the spacetime interval, *s*) that is a combination of distance and time that does not change; it is the same for all observers.

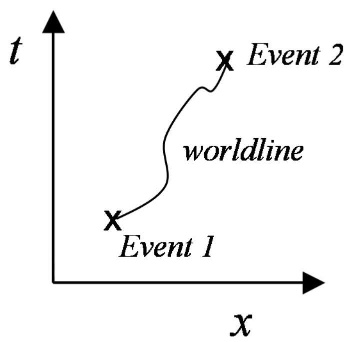
This is why we need 4-D spacetime: to plot this new 'length' in. Time becomes another axis. The difference is that it has to be multiplied by *c* and its square comes into the definition of the interval as opposite in sign to the spatial dimensions. Here is how time must be regarded as the 'fourth dimension'.

Einstein's old teacher, Herman Minkowski, realised this in 1908. He said:
«Henceforth, space by itself and time by itself are doomed to fade away into mere shadows and only a kind of union of the two will preserve an independent reality.»

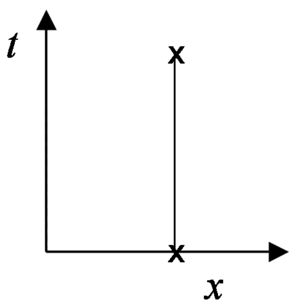
So, lengths and times vary from one frame to the next, but this is because see them separately. Viewed in 4-D spacetime, it is like looking at a length (the interval) from different angles (perspectives). To an observer in one frame there may be of a spatial component and less of a time component. To another, it might be the other way round. But combining space and time means the spacetime interval in 4-D is constant.

Spacetime diagrams

Everything that happens in the Universe is just a collection of events. These trace continuous loci (paths) in 4D spacetime, which will look different in different reference frames. They are pictured in a 'spacetime diagram'. Of course, we cannot plot four mutually perpendicular axes and so, to simplify, we forget about the *y* and *z* axes and plot just *x, t*.



If an object starts at x_1 at time t_1 and moves (at varying speed) until it reaches x_2 at t_2, then the line joining the two points is the object's worldline. Everythings traces worldlines through spacetime. You have a worldline that is made up from the sum of worldlines of all the atoms in your body. Many will merge together when you are born and this clump of worldlines makes up yours. Many atoms will leave off on their own worldline as you shed them while trillions of others join throughout your lifetime.



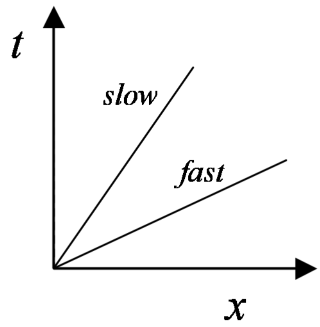
An object even traces a worldline in spacetime when it is standing still (see left).

The faster an object moves the bigger the angle its worldline makes with the time axis.

The gradient of a worldline here is the reciprocal of its velocity since gradient here is 1/*v.*

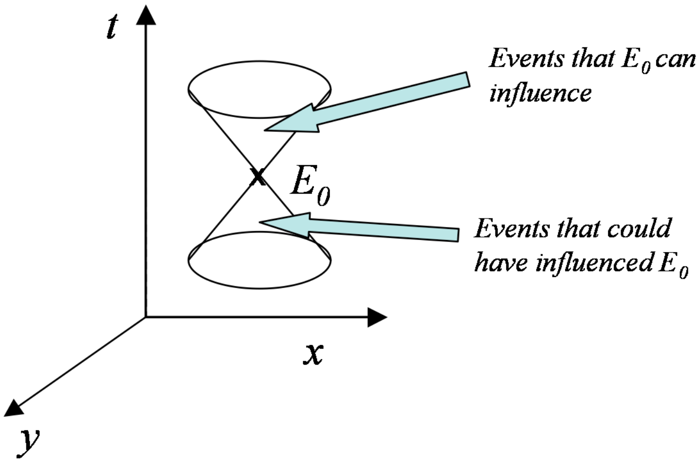
Since the fastest speed in the Universe is the speed of light (we will see why in the next lecture) there is a certain minimum gradient that worldlines cannot drop under.

We define the our axes scales such that the worldline of a light beam makes 45°.



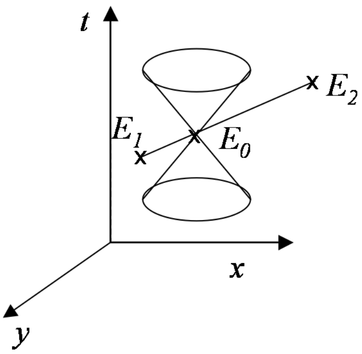
Since nothing can go faster than light, no body can have a worldline of more than 45° with the time axis.

<u>Light cones</u>

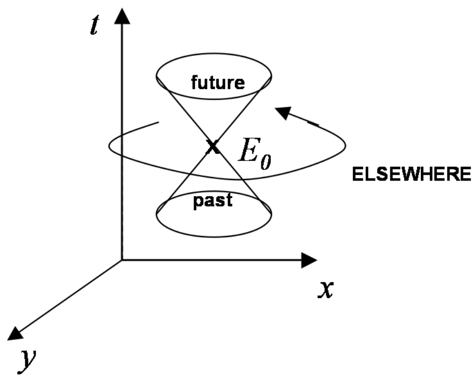


Putting back in the *y*-axis allows us to draw what is called a light cone. In fact, every event is associated with two light cones, one below and one, inverted, above. All events sitting in the light cone below E_0 could have influenced it or caused it since we can plot a worldline between any of these events and E_0 that has a gradient steeper than 45°. Likewise, any events in light cone above E_0 could have been caused by it.

Any lines joining E_0 with events outside its light cone would necessarily imply faster than light travel if we think they are causally connected. Thus, in the diagram below, while E_1 has happened, in this frame, before E_0 and E_1 has happened after, neither is casally linked to it as the distance that would need to be covered is greater than be covered by light in the short time interval between them.



The two light cones are known as future and past light cones and everywhere outside of them for any given event in a given frame is known as ‘elsewhere’.



Now go back to 2D spacetime and we see that on the surface of the light cone (well, light triangle now) we have $x = ct$ where $c = 1$ is put in for the sake of correct units.

Thus, the intervals between any two events on the this line are related via

$$\Delta x^2 = c^2 \Delta t^2$$

and in full 4-D spacetime this is

$$\Delta x^2 + \Delta y^2 + \Delta z^2 = c^2 \Delta t^2 \qquad \Rightarrow s^2 = 0$$

Inside the cone:

If s^2 is **positive** then $c^2 \Delta t^2 > \Delta x^2 + \Delta y^2 + \Delta z^2$ and such an interval is known as **timelike.**

If s^2 is **negative** then $c^2 \Delta t^2 < \Delta x^2 + \Delta y^2 + \Delta z^2$ and such an interval is known as **spacelike** and will only join events if there is ftl signalling!