The Lorentz Transformation Equations

This set of equations, derived by Lorentz before 1905, show how we must transform from one inertial reference frame to another and they replace the old Newtonian/Galilean Transformation Equations. They will not be derived here, simply stated. But the interested student can find their derivation in any textbook on special relativity.

For two inertial frames of reference, *S* and *S′*, in standard configuration (i.e with *S′* moving at a constant speed *v* relative to *S* in the positive *x*-direction):

where $\gamma = 1 \Big/ \sqrt{1 - v^2/c^2}$

$x' = \gamma(x - vt)$	$x = \gamma(x' + vt)$
$y' = y$	$y = y'$
$z' = z$	$z = z'$
$t' = \gamma(t - vx/c^2)$	$t = \gamma(t' + vx'/c^2)$

And the distance and interval between two events (1) and (2) are:

$\Delta x' = \gamma(\Delta x - v\Delta t)$	$\Delta x = \gamma(\Delta x' + v\Delta t')$
$\Delta y' = \Delta y$	$\Delta y = \Delta y'$
$\Delta z' = \Delta z$	$\Delta z = \Delta z'$
$\Delta t' = \gamma(\Delta t - v\Delta x / c^2)$	$\Delta t = \gamma(\Delta t' + v\Delta x' / c^2)$

You must take care in using these transformation equations. The first set (the upper box) are to calculate spacetime coordinates of events in different frames. The more useful ones are the second set – the 'difference' form of the L.T. equations – which calculate the lengths ($\Delta x, \Delta x'$) of objects, or distance between events, in different frames, as well as the rates that clocks tick, or intervals between events, in different frames. We need, however, to define two important concepts:

Proper length: This is the length of an object as measured in the frame in which the object is at rest.

Proper time: a) This is the time difference between two events happening at the same place (not spatially separated) as measured by a clock in the same frame.

b) Another way of stating this is that it is the time measured by a clock as seen by an observer in the clock's rest frame (ie for whom the clock is at rest).

Two consequences of the L.T. equations that we will investigate are time dilation (the slowing down of time) and length contraction (also called Lorentz contraction as he was the first to predict the effect, albeit for the wrong reason). This contraction in length means objects appear shorter when they are moving very fast. It also means that the distance travelled by a fast moving object appears, in the rest frame of that object, to be a shorter distance. That is, if a rocket that is travelling from A to B that are 10 lightyears apart as measured in the rest frame of A and B, the distance needed to cover will appear shorter according to the astronauts on board the rocket. This is because, in the moving frame, the distance (the space) that has to be covered that appears to moving in the opposite direction. Think of a ruler stretch between A and B. It will appear shorter as seen from the rocket since it is a moving object.

Example:

Time dilation.

a) for a clock at rest in a spaceship (frame *S'*) observed at two different times separated by an interval $\Delta t'$ as measured by the spaceship clock. What is the time between the events as measured by the Earth clock (frame *S*)?

We need to use $\Delta t = \gamma(\Delta t' + v\Delta x' / c^2)$.

The distance between the two events as measured in *S'* is $\Delta x' = 0$ (since the clock has not moved). So we are left with

$$\Delta t = \gamma\, \Delta t'$$

which agrees with our earlier formula from light pulse clock.

So interval between the two events (two ticks of the rocket clock) that is one second long will take longer according to Earth clocks – rocket clock running slower.

b) For clock on Earth at two times, $\Delta x = 0$, which means that we must use

$$\Delta t' = \gamma(\Delta t - v\Delta x / c^2) = \gamma\, \Delta t$$

So here we see that, seen from rocket, it is the Earth clock that is ticking more slowly! Is this a paradox? No everything is relative. We need to wait till the discussion of the twins paradox to see this more closely.

Length contraction:

Viewed from *S* what is the length of something that is stationary in *S'* ? That is an object moving at velocity *v* relative to observer stationary in *S*.
An important point here is that, in *S'*, the length $\Delta x'$ is the proper length.

Be careful here how to use the L.T. equation you need to use. You must think about what quantities are known. Thus, since we need Δx (length as measure in *S*) we cannot check positions of each end of the moving object at different times; there positions must be known simultaneously (i.e. a snapshot of the moving object). This means there is no time interval Δt in *S.* Thus $\Delta t = 0$.

So problem involves Δx $\Delta x'$ and Δt . So we need

$$\Delta x' = \gamma(\Delta x - v\Delta t)$$
$$\Rightarrow \Delta x' = \gamma \Delta x$$
$$\boxed{\Rightarrow \Delta x = \frac{1}{\gamma}\Delta x'}$$

and since $\gamma > 1$, this means that $\Delta x > \Delta x'$.

Example with numbers:

An aircraft, of length 40m at rest, is moving at 630m/s with respect to the ground. By how much will its length appear to be contracted according to an observer on the ground?

From the above boxed equation,

$$\Delta x = \sqrt{1-\frac{v^2}{c^2}} \times 40 = \sqrt{1-\left(\frac{630}{3\times 10^8}\right)^2} \times 40 = \sqrt{1-4.4\times 10^{-12}} \times 40$$

Thus length is contract by

$$\Delta x' - \Delta x = 40 - \sqrt{1-4.4\times 10^{-12}} \times 40 = 40\left(1-\sqrt{1-4.4\times 10^{-12}}\right)$$

Here we must use the Binomial approximation since just pushing buttons on the caclulator to work out above expression will give you zero (try it!)

We use the fact that $(1-x)^n \simeq 1-nx,$ when x is very small.

Applying this gives us the result $\Delta x' - \Delta x = 8.8\times 10^{-11}$ = 0.88 Å (angstroms). That is, less than the size of an atom.

Now back to time dilation: How long will it take, by Earth clocks, for the aircraft clock to fall behind by 1μs?

We want $\Delta t'$ to be less than Δt by 1μs. Here $\Delta t'$ is the proper time and we just need the time dilation relation since there are no lengths involved: $\Delta t = \gamma \Delta t'$. Of course what is the subtlety here is that we know neither Δt nor $\Delta t'$, but their difference:

$$\Delta t - \Delta t' = 1\times 10^{-6} = \Delta t - \frac{1}{\gamma}\Delta t$$
$$\Rightarrow \Delta t = \frac{1\times 10^{-6}}{(1-1/\gamma)} = \frac{1\times 10^{-6}}{2.2\times 10^{-12}}$$

The above quantity is in seconds, so converting to more interesting units this gives

$\Delta t \approx$ 5 days.

Velocity transformations:

So far we have discussed bodies at rest in one of the two frames. The moving body's rest frame is usually chosen to be *S′*. What if the two frames are already defined and one is moving relative to the other? We can then have the situation of a body moving relative to an observedr in *S′*, and we can ask what its velocity will be according to a second observer in *S.*

Consider a body moving at velocity v_x in frame *S.* What is its velocity in frame *S′* that is itself moving at a velocity v_0 relative to *S*?

We can derive a new set of velocity transformation equations from the L.T. equations. [See tutorial sheet problem.]

Note that in general, the body may be moving in some arbitrary direction in *S* and therefore have components v_x, v_y, v_z

$$v'_x = \frac{v_x - v_0}{1 - v_x v_0 / c^2} \qquad v_x = \frac{v'_x + v_0}{1 + v'_x v_0 / c^2}$$

$$v'_y = \frac{v_y}{\gamma\left(1 - v_x v_0 / c^2\right)} \qquad v_y = \frac{v'_y}{\gamma\left(1 + v'_x v_0 / c^2\right)}$$

$$v'_z = \frac{v_z}{\gamma\left(1 - v_x v_0 / c^2\right)} \qquad v_z = \frac{v'_z}{\gamma\left(1 + v'_x v_0 / c^2\right)}$$

Note that:

1. If v_x and v_0 are both $\ll c$ then we obtain $v'_x = v_x - v_0$ (Galilean Transformation)
2. If $v_x = c$ then

$$v'_x = \frac{c - v_0}{1 - {v_0}/{c}} = \frac{c - v_0}{c - v_0} \times c = c$$

and in agreement with Einstein's second Postulate about the constancy of the speed of light.

3. If a body is moving in the y or z direction then we do not get the simple $v'_y = v_y$ or $v'_z = v_z$. Consider it is moving in the y-direction, then $v_x = v_z = 0$

Therefore,

$$v'_y = \frac{v_y}{\gamma(1-0)} = v_y / \gamma$$

That is, the velocity appears dilated (reduced) by a factor of γ when viewed in the *S′* frame.

Example:
(see cartoon in *Black Holes, Wormholes and Time Machines*)
A space traveller in a rocket moving in the *x*-direction at a speed of $v_0 = 0.5c$ relative to an observer (at rest in frame *S*) fires a bullet in the same direction of motion of the rocket from his super gun. It leaves the gun at $v'_x = 0.8c$. How fast does the bullet travel according to the observer in *S*?

Note that if we simply add velocities here we get 0.8c+0.5c=1.3c (faster than light!)

In fact we require the transformation equation:

$$
\begin{aligned}
v_x &= \frac{v'_x + v_0}{1 + v'_x v_0 / c^2} \\
&= \frac{0.8c + 0.5c}{1 + 0.8c \times 0.5c / c^2} = \frac{1.3c}{1.4} = 0.93c
\end{aligned}
$$

That is, still less than *c*. As we saw earlier, even if bullet speed were equal to *c* then observer would still see it moving at *c.*